# Inapproximability of the Tutte polynomial*

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February 1, 2008

**Abstract**

The Tutte polynomial of a graph $G$ is a two-variable polynomial $T(G; x, y)$ that encodes many interesting properties of the graph. We study the complexity of the following problem, for rationals $x$ and $y$: take as input a graph $G$, and output a value which is a good approximation to $T(G; x, y)$. Jaeger, Vertigan and Welsh have completely mapped the complexity of *exactly* computing the Tutte polynomial. They have shown that this is #P-hard, except along the hyperbola $(x-1)(y-1) = 1$ and at four special points. We are interested in determining for which points $(x, y)$ there is a *fully polynomial randomised approximation scheme* (FPRAS) for $T(G; x, y)$. Under the assumption RP $\neq$ NP, we prove that there is no FPRAS at $(x, y)$ if $(x, y)$ is is in one of the half-planes $x < -1$ or $y < -1$ (excluding the easy-to-compute cases mentioned above). Two exceptions to this result are the half-line $x < -1, y = 1$ (which is still open) and the portion of the hyperbola $(x-1)(y-1) = 2$ corresponding to $y < -1$ which we show to be equivalent in difficulty to approximately counting perfect matchings. We give further intractability results for $(x, y)$ in the vicinity of the origin. A corollary of our results is that, under the assumption RP $\neq$ NP, there is no FPRAS at the point $(x, y) = (0, 1-\lambda)$ when $\lambda > 2$ is a positive integer. Thus, there is no FPRAS for counting nowhere-zero $\lambda$ flows for $\lambda > 2$. This is an interesting consequence of our work since the corresponding decision problem is in P for example for $\lambda = 6$. Although our main concern is to distinguish regions of the

*Partially supported by the EPSRC grant Discontinuous Behaviour in the Complexity of Randomized Algorithms. A preliminary version of this paper appeared in the proceedings of the ACM *Symposium on Theory of Computing* [9]

Tutte plane that admit an FPRAS from those that do not, we also note that the latter regions exhibit different levels of intractability. At certain points $(x, y)$, for example the integer points on the $x$-axis, or any point in the positive quadrant, there is a randomised approximation scheme for $T(G; x, y)$ that runs in polynomial time using an oracle for an NP predicate. On the other hand, we identify a region of points $(x, y)$ at which even approximating $T(G; x, y)$ is as hard as #P.

## 1 Summary of results

The Tutte polynomial of a graph $G$ (see Section 2.1) is a two-variable polynomial $T(G; x, y)$ that encodes many interesting properties of the graph. We mention only some of these properties here, as a longer and more detailed list can be found in Welsh's book [22].

- $T(G; 1, 1)$ counts the number of spanning trees of a connected graph $G$.
- $T(G; 2, 1)$ counts the number of forests in $G$ (the number of edge subsets that contain no cycles).
- $T(G; 1, 2)$ counts the number of edge subsets that are connected and span $G$.
- $T(G; 2, 0)$ counts the number of acyclic orientations of $G$.
- The chromatic polynomial $P(G; \lambda)$ of a graph $G$ with $n$ vertices, $m$ edges and $k$ connected components is given by

  $$P(G; \lambda) = (-1)^{n-k}\lambda^k T(G; 1-\lambda, 0).$$

  When $\lambda$ is a positive integer, $P(G; \lambda)$ counts the proper $\lambda$-colourings of $G$.
- The flow polynomial $F(G; \lambda)$ is given by

  $$F(G; \lambda) = (-1)^{m-n+k} T(G; 0, 1-\lambda).$$

  When $\lambda$ is a positive integer, $F(G; \lambda)$ counts the nowhere-zero $\lambda$-flows of $G$.
- The all-terminal reliability polynomial $R(G; p)$ is given by

  $$R(G; p) = (1-p)^{m-n+k} p^{n-k} T(G; 1, 1/(1-p)).$$

  When $G$ is connected and each edge is independently "open" with probability $p$, $R(G, p)$ is the probability that there is a path between every pair of vertices of $G$.

- For every positive integer $q$, the Tutte polynomial along the hyperbola $H_q$ given by $(x-1)(y-1) = q$ corresponds to the partition function of the $q$-state Potts model.

We study the complexity of the following problem, for rationals $x$ and $y$: take as input a graph $G$, and output a value which is a good approximation to $T(G; x, y)$. Jaeger, Vertigan and Welsh [10] (see Section 2.4) have completely mapped the complexity of *exactly* computing the Tutte polynomial. They have shown that this is #P-hard, except along the hyperbola $H_1$ and at the four special points $(x, y) \in \{(1, 1), (0, -1), (-1, 0), (-1, -1)\}$. (#P is the analogue, for counting problems, of the more familiar class NP of decision problems.)

We are interested in determining for which points $(x, y)$ there is a *fully polynomial randomised approximation scheme* (FPRAS) for $T(G; x, y)$. An FPRAS is a polynomial-time randomised approximation algorithm achieving arbitrarily small relative error. Precise definitions of FPRAS, #P, and other complexity-theoretic terminology will be provided in Section 2.2.

It is known that there is an FPRAS for every point $(x, y)$ on the hyperbola $H_2$ with $y > 1$ — this follows from the Ising result of Jerrum and Sinclair [12]. No other general FPRAS results are known. A few negative results are known — see Section 2.4.

Our goal is to map the Tutte plane in terms of FPRASability as completely as possible. The specific contribution of this article is a substantial widening of the region known to be non-FPRASable.

Our contributions are summarised in Figure 1. In particular, under the assumption RP $\neq$ NP, we prove the following.

(1) If $x < -1$ and $(x, y)$ is not on $H_0$ or $H_1$, then there is no FPRAS at $(x, y)$ (Corollary 4).

(2) If $y < -1$ and $(x, y)$ is not on $H_1$ or $H_2$, then there is no FPRAS at $(x, y)$ (Corollary 5 when $(x, y)$ is not on $H_0$ and Lemma 6 for the case in which $(x, y)$ is on $H_0$).

(3) If $(x, y)$ is on $H_2$ and $y < -1$ then approximating $T(G; x, y)$ is equivalent in difficulty to approximately counting perfect matchings (Lemma 7).

(4) If $(x, y)$ is not on $H_1$ and is in the vicinity of the origin in the sense that $|x| < 1$ and $|y| < 1$ and is in the triangle given by $y < -1 - 2x$ then there is no FPRAS (Lemma 8).

(5) If $(x, y)$ is not on $H_1$ and is in the vicinity of the origin and is in the triangle given by $x < -1 - 2y$ then there is no FPRAS (Lemma 9).

(6) The two previous intractability results (results (4) and (5)) can be partially extended to the boundary of the triangles (Lemma 10 and 11).

(7) If $(x, y)$ is in the vicinity of the origin and $q = (x-1)(y-1) > 1.5$ then there is no FPRAS (excluding the special points at which exact computation is possible) (Lemma 12).

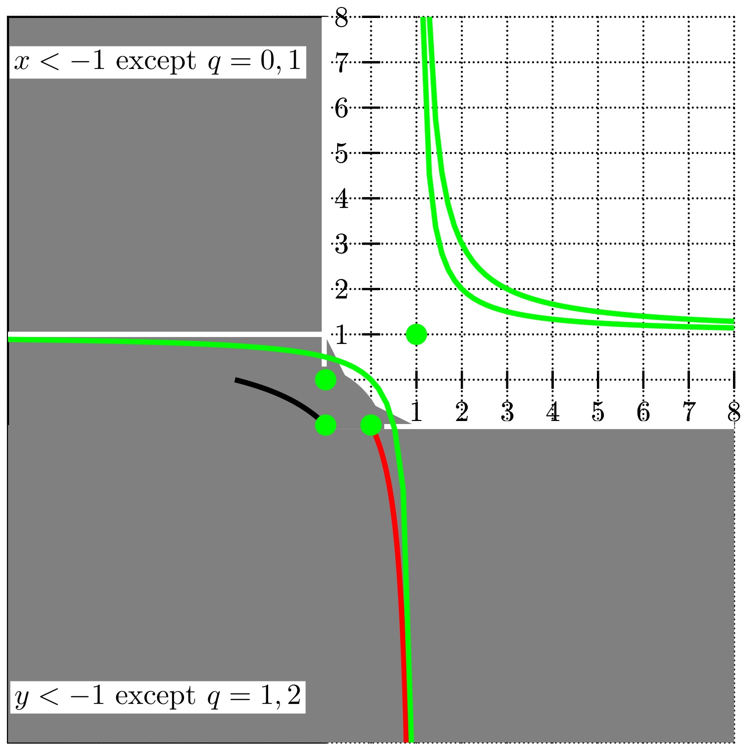


Figure 1: Green points are FPRASable, red points are equivalent to perfect matchings and gray points are not FPRASable unless RP=NP. We don't know about white points. The points depicted in black are at least as hard as gray and are presumably harder — this is the region of $q = 4$ with $y \in (-1, 0)$ and approximating Tutte is actually #P-hard here. (There are presumably more such points.)

Result (2) above implies that, under the assumption RP $\neq$ NP, there is no FPRAS at the point $(x, y) = (0, 1-\lambda)$ when $\lambda > 2$ is a positive integer. Thus, there is no FPRAS for counting nowhere-zero $\lambda$ flows for $\lambda > 2$. This is an interesting consequence of our work since Seymour [16] has shown that the corresponding decision problem is in P for $\lambda = 6$. In particular, a graph has a 6-flow if and only if it has no bridge (cut edge).

Although our main concern is to distinguish regions of the Tutte plane that admit an FPRAS from those that do not, we also note that the latter regions exhibit different levels of intractability. At certain points $(x, y)$, for example the integer points on the $x$-axis, or any point in the positive quadrant, there is a randomised approximation scheme for $\mathrm{T{\scriptsize UTTE}}(x, y)$ that runs in polynomial time using an oracle for an NP predicate. On the other hand,

Theorem 17 identifies a region of points $(x, y)$ at which even approximating $\text{TUTTE}(x, y)$ is as hard as #P. These two kinds of intractability are very different, assuming #P is a "much bigger" class than NP.

# 2 Definitions and context

## 2.1 The Tutte polynomial

The Tutte polynomial of a graph $G = (V, E)$ is the two-variable polynomial

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{\kappa(A)-\kappa(E)} (y-1)^{|A|-n+\kappa(A)}, \tag{1}$$

where $\kappa(A)$ denotes the number of connected components of the graph $(V, A)$ and $n = |V|$. Following the usual convention for the Tutte polynomial [17] a graph is allowed to have loops and/or multiple edges, and we use the term "graph" in this way except where we explicitly state otherwise. The Tutte polynomial is sometimes referred to as the "Whitney-Tutte" polynomial, or the "dichromatic polynomial". See [20] and [22].

## 2.2 The complexity of counting and approximate counting

We start with a brief summary of the complexity of counting. See [11] for more details. A counting problem can be viewed as a function $f : \Sigma^* \to \mathbb{N}$ mapping an encoding of a problem instance (encoded as a word in a finite alphabet, $\Sigma$) to a natural number. For example, $f$ might map an encoding of a graph $G$ to the number of independent sets of $G$. #P is the analogue of NP for counting problems. A counting problem $f : \Sigma^* \to \mathbb{N}$ is in #P if there is a polynomial-time predicate $\chi : \Sigma^* \times \Sigma^* \to \{0, 1\}$ and a polynomial $p$ such that, for all instances $x \in \Sigma^*$,

$$f(x) = |\{w \in \Sigma^* \mid \chi(x, w) \wedge |w| \leq p(|x|)\}|.$$

It is straightforward to check that $\text{TUTTE}(x, y) \in$ #P when $x, y$ are integers with $x, y \geq 1$. If $x, y$ are arbitrary integers then the terms in the Tutte polynomial vary in sign, and the problem $\text{TUTTE}(x, y)$ no longer fits the #P framework. In that case, $\text{TUTTE}(x, y) \in$ GapP, where GapP is the set of functions $f = f^+ - f^- : \Sigma^* \to \mathbb{Z}$ expressible as the difference of two #P-functions $f^+$ and $f^-$. (Simply partition the terms of the Tutte polynomial according to sign, and compute the positive and negative parts separately.)

Finally, since we do not want to restrict ourselves to integer $x$ and $y$, we need to extend the classes #P and GapP a little to encompass computations over the rationals. We say that $f : \Sigma^* \to \mathbb{Q}$ is in the class $\#\text{P}_{\mathbb{Q}}$ if $f(x) = a(x)/b(x)$, where $a, b : \Sigma^* \to \mathbb{N}$, and $a \in$ #P and $b \in$ FP, where FP is the

class of functions computable by polynomial-time algorithms. If $x, y \geq 1$, then $\textsc{Tutte}(x, y) \in \#\mathrm{P}_{\mathbb{Q}}$, since we may multiply through by suitable powers of the denominators of $x$ and $y$, after which all the terms in the Tutte polynomial become integers.

A *randomised approximation scheme* is an algorithm for approximately computing the value of a function $f : \Sigma^* \to \mathbb{R}$. The approximation scheme has a parameter $\varepsilon > 0$ which specifies the error tolerance. A *randomised approximation scheme* for $f$ is a randomised algorithm that takes as input an instance $x \in \Sigma^*$ (e.g., an encoding of a graph $G$) and an error tolerance $\varepsilon > 0$, and outputs a number $z \in \mathbb{Q}$ (a random variable of the "coin tosses" made by the algorithm) such that, for every instance $x$,

$$\Pr\left[e^{-\varepsilon} f(x) \leq z \leq e^{\varepsilon} f(x)\right] \geq \frac{3}{4}. \tag{2}$$

The randomised approximation scheme is said to be a *fully polynomial randomised approximation scheme*, or *FPRAS*, if it runs in time bounded by a polynomial in $|x|$ and $\varepsilon^{-1}$. Note that the quantity 3/4 in Equation (2) could be changed to any value in the open interval $(\frac{1}{2}, 1)$ without changing the set of problems that have randomised approximation schemes [13, Lemma 6.1].

It is known that every counting problem in #P has a randomised approximation scheme whose complexity is not much greater than NP. In particular, if $f$ is a counting problem in #P then the bisection technique of Valiant and Vazirani [21, Cor 3.6] can be used to construct a randomised approximation scheme for $f$ that runs in polynomial time, using an oracle for an NP predicate. See [13, Theorem 3.4] or [5, Theorem 1]; also [18] for an early result in this direction.

We will use the notion of approximation-preserving reduction from Dyer, Goldberg, Greenhill and Jerrum [5]. Suppose that $f$ and $g$ are functions from $\Sigma^*$ to $\mathbb{R}$. An "approximation-preserving reduction" from $f$ to $g$ gives a way to turn an FPRAS for $g$ into an FPRAS for $f$. An *approximation-preserving reduction* from $f$ to $g$ is a randomised algorithm $\mathcal{A}$ for computing $f$ using an oracle for $g$. The algorithm $\mathcal{A}$ takes as input a pair $(x, \varepsilon) \in \Sigma^* \times (0, 1)$, and satisfies the following three conditions: (i) every oracle call made by $\mathcal{A}$ is of the form $(w, \delta)$, where $w \in \Sigma^*$ is an instance of $g$, and $0 < \delta < 1$ is an error bound satisfying $\delta^{-1} \leq \mathrm{poly}(|x|, \varepsilon^{-1})$; (ii) the algorithm $\mathcal{A}$ meets the specification for being a randomised approximation scheme for $f$ (as described above) whenever the oracle meets the specification for being a randomised approximation scheme for $g$; and (iii) the run-time of $\mathcal{A}$ is polynomial in $|x|$ and $\varepsilon^{-1}$.

If an approximation-preserving reduction from $f$ to $g$ exists we write $f \leq_{\mathrm{AP}} g$, and say that *$f$ is AP-reducible to $g$*. Note that if $f \leq_{\mathrm{AP}} g$ and $g$ has an FPRAS then $f$ has an FPRAS. (The definition of AP-reduction was chosen to make this true). If $f \leq_{\mathrm{AP}} g$ and $g \leq_{\mathrm{AP}} f$ then we say that *$f$ and $g$ are AP-interreducible*, and write $f \equiv_{\mathrm{AP}} g$.

Dyer et al. [5] identified three classes of counting problems that are interreducible under approximation-preserving reductions. The first class, containing the problems that admit an FPRAS, are trivially AP-interreducible since all the work can be embedded into the reduction (which declines to use the oracle). The second class (and the last one that we will describe here) is the set of problems that are AP-interreducible with #SAT, the problem of counting satisfying assignments to a Boolean formula in CNF. Zuckerman [24] has shown that #SAT cannot have an FPRAS unless RP = NP. The same is obviously true of any problem in #P to which #SAT is AP-reducible. See [5] for details.

### 2.3 The Tutte polynomial and #P

We will study the following computational problem for rationals $x$ and $y$.

**Name** TUTTE$(x, y)$.

**Instance** A graph $G = (V, E)$.

**Output** $T(G; x, y)$.

Given fixed rationals $x$ and $y$, TUTTE$(x, y)$ is a function from $\Sigma^*$ to $\mathbb{Q}$, mapping an encoding of a graph $G$ to a rational $T(G; x, y)$. It is not immediately clear from the definition (1) that TUTTE$(x, y)$ is in $\#\mathrm{P}_{\mathbb{Q}}$, but this is known to be true if $x$ and $y$ are both non-negative.

In particular, if $G$ is connected, it is known [19] (see also [20, Theorem 1X.65]) that $T(G; x, y)$ can be expressed as $T(G; x, y) = \sum_T x^{a(T)} y^{b(T)}$, where the sum is over spanning trees $T$ of $G$, and $a(T)$ and $b(T)$ are natural numbers which are easily computable from $T$.[1] ($a(T)$ is the number of so-called "internally active" edges of $T$ and $b(T)$ is the number of "externally active" edges of $T$ — see [20] for details.)

It is clear from the definition (1) that the Tutte polynomial of a graph $G$ (which may have several connected components) may be expressed as a product of the Tutte polynomials of the components. Thus, for $x \geq 0$ and $y \geq 0$, we have TUTTE$(x, y) \in \#\mathrm{P}_{\mathbb{Q}}$, which means that there is a randomised approximation scheme for TUTTE$(x, y)$ that runs in polynomial time, using an oracle for an NP predicate.

It is unlikely that TUTTE$(x, y)$ is in $\#\mathrm{P}_{\mathbb{Q}}$ for all $x$ and $y$. In particular, Theorem 17 identifies a region of points $(x, y)$, where $y$ is negative, for which even approximating TUTTE$(x, y)$ is as hard as #P.

### 2.4 Previous work on the complexity of the Tutte polynomial

Jaeger, Vertigan and Welsh [10] have completely mapped the complexity of exactly computing the Tutte polynomial. They have observed that

[1] Indeed, historically, this appears to have been the original definition of the polynomial.

$\text{TUTTE}(x, y)$ is in FP for any point $(x, y)$ on the hyperbola $H_1$. This can be seen from the definition (1), since terms involving $\kappa(A)$ cancel. Also, $\text{TUTTE}(x, y)$ is in FP when $(x, y)$ is one of the special points $(1, 1)$, $(0, -1)$, $(-1, 0)$, and $(-1, -1)$. As noted in Section 1, $T(G; 1, 1)$ is the number of spanning trees of a connected graph $G$, $T(G; 0, -1)$ is the number of 2-flows of $G$ (up to a factor of plus or minus one), and $T(G; -1, 0)$ is the number of 2-colourings of $G$ (up to an easily computable factor). $T(G; -1, -1)$ has an interpretation in terms of the "bicycle space" of $G$. See [10, (2.8)]. Intriguingly, Jaeger, Vertigan and Welsh managed to show that $\text{TUTTE}(x, y)$ is #P-hard for every other pair of rationals $(x, y)$. They also investigated the complexity of evaluating the Tutte polynomial when $x$ and $y$ are real or complex numbers, but that is beyond the scope of this paper.

The only FPRAS for approximating the Tutte polynomial that we know of is the Ferromagnetic Ising FPRAS of Jerrum and Sinclair [12]. This gives an FPRAS for $\text{TUTTE}(x, y)$ for every point $(x, y)$ on $H_2$ with $y > 1$. The connection between the Ising model and the Tutte polynomial along the hyperbola $H_2$ is elaborated later in the paper — see Equation (39). We know of no other FPRASes for approximating the Tutte polynomial for an arbitrary input graph $G$. There is some related work, however, for example, Karger [14] gives an FPRAS for **non**-reliability, which is not the same thing as an FPRAS for reliability, but is somewhat related.

There are also FPRASes known for special cases in which restrictions are placed on $G$. For example, [1] gives an FPRAS for points $(x, y)$ with $x \geq 1$ and $y \geq 1$ for the restricted case in which the input graph $G$ is "dense", meaning that the $n$-vertex graph $G$ has minimum degree $\Omega(n)$. As another example, there is a huge literature on approximately counting proper colourings of degree-bounded graphs.

Several negative results are known for approximating the Tutte polynomial. First, note that if $T(G; x, y)$ is the number of solutions to an NP-complete decision problem, then there can be no FPRAS for $\text{TUTTE}(x, y)$ unless RP = NP. So, for example, if $\lambda > 2$ is a positive integer, then by the chromatic polynomial specialisation mentioned in Section 1, there is no FPRAS for $T(G; 1 - \lambda, 0)$.

Jerrum and Sinclair [12, Theorem 14] showed that there is no FPRAS for (antiferromagnetic) Ising unless RP = NP. This implies that, unless RP = NP, there is no FPRAS for the function whose input is a graph $G$ and a point $(x, y)$ on $H_2$ with $0 < y < 1$ and whose output is $T(G; x, y)$.

Welsh [23] extended this result. Specifically, he showed the following, assuming RP $\neq$ NP.

- Suppose $q \geq 2$ is a positive integer. Then there is no FPRAS for the function whose input is a graph $G$ and a point $(x, y)$ on $H_q$ with $x < 0, y > 0$ and whose output is $T(G; x, y)$. Furthermore, there is no FPRAS for the function whose input is a graph $G$ and a point $(x, y)$

on $H_q$ with $x < 0, y < 0$ and whose output is $T(G; x, y)$.

- There is no FPRAS for the function whose input is a graph $G$ and a point $(x, y)$ on $H_3$ with $0 < x < 1$ and whose output is $T(G; x, y)$.

# 3 Regions of the Tutte plane that do not admit an FPRAS unless RP=NP

The tensor product of matroids was introduced by Brylawski [2]. We define it here in the special case of graphs. Let $G$ be a graph, and $K$ another graph with a distinguished edge $e$ with endpoints $u$ and $u'$. The tensor product $G \otimes K$ is obtained from $G$ by performing a 2-*sum* operation with $K$ on each edge $f$ of $G$ in turn: Let the endpoints of $f$ be $v$ and $v'$. Take a copy of $K$ and identify vertex $u$ (resp. $u'$) of $K$ with $v$ (resp. $v'$) of $G$, and then delete edges $e$ and $f$. (Since $G$ and $K$ are undirected graphs, there are two ways of performing the 2-sum. This lack of uniqueness is an artefact of viewing a matroid operation in terms of graphs, which have additional structure. However, the Tutte polynomial is insensitive to which of the two possible identifications is made.) For technical reasons we will assume that $e$ is not a bridge of $K$. In particular, we assume that deleting $e$ does not increase the number of connected components of $K$.

Let $K \backslash e$ be the graph constructed from $K$ by deleting edge $e$. Let $K/e$ be the graph constructed from $K$ by contracting edge $e$. Suppose $(x, y) \in \mathbb{Q}^2$. Let $q = (x-1)(y-1)$. Define the point $(x', y')$ as follows.

$$x' = \frac{(1-q)T(K \setminus e; x, y)}{T(K \setminus e; x, y) - (x-1)\,T(K/e; x, y)} \tag{3}$$

and

$$y' = \frac{(1-q)T(K/e; x, y)}{T(K/e; x, y) - (y-1)T(K \setminus e; x, y)}. \tag{4}$$

Then it is known ([10, (4.1)]) that

$$T(G; x', y') = L(x, y, K)^m M(x, y, K)^{n-\kappa} T(G \otimes K; x, y), \tag{5}$$

where $n$, $m$ and $\kappa$ are (respectively) the number of vertices, edges and connected components in $G$ and

$$L(x, y, K) = \frac{1-q}{T(K/e; x, y) - (y-1)T(K \setminus e; x, y)},$$

and

$$M(x, y, K) = \frac{T(K/e; x, y) - (y-1)T(K \setminus e; x, y)}{T(K \setminus e; x, y) - (x-1)\,T(K/e; x, y)}.$$

Suppose that the denominators of (3) and (4) are non-zero. In this case, the point $(x',y')$ is well-defined and we say that $(x,y)$ is *shifted to* the point $(x',y')$ by $K$. In this case, $L(x,y,K)$ and $M(x,y,K)$ are also well-defined, so Equation (5) gives us the reduction $\textsc{Tutte}(x',y') \leq_{\mathrm{AP}} \textsc{Tutte}(x,y)$.

We will not prove Equation (5) since the equation can be found elsewhere (eg., [10, (4.1)]) but, for completeness, we will derive similar identities that we will use in Section 4. We are particularly interested in two special cases. The case in which $K$ is a cycle on $k+1$ vertices is known as a *$k$-stretch* in the literature and the case in which $K$ is a two-vertex graph with $k+1$ parallel edges is known as a *$k$-thickening*. Informally, a $k$-stretch of $G$ replaces each edge of $G$ by a path of length $k$, while a $k$-thickening replaces each edge by a bundle of $k$ parallel edges. Specifically,

$$(x',y') = \begin{cases} \big(x^k, q/(x^k-1)+1\big) & \text{for a } k\text{-stretch;} \\ \big(q/(y^k-1)+1, y^k\big) & \text{for a } k\text{-thickening.} \end{cases} \tag{6}$$

Observe that $q=(x-1)(y-1)$ is an invariant for stretches and thickenings, and indeed for shifts in general. It is this limitation that gives the hyperbolas $H_q$ a special place in the complexity theory of the Tutte polynomial. All shifts preserve $q=(x-1)(y-1)$ but not all AP-reductions do. In particular, the construction in [10, (5.12)] (taking $p=1$), based on an idea of Linial [15], gives the reduction $\textsc{Tutte}(x,0) \leq_{\mathrm{AP}} \textsc{Tutte}(x-1,0)$ for $x \neq 1$.

We shall make frequent use of the fact that shifts may be composed.

**Lemma 1.** *The relation "shifts to" is transitive.*

*Proof.* Suppose $K_1$ is a graph that implements the shift $(x,y) \to (x',y')$ and $K_2$ is the graph (with distinguished edge $e$) that implements $(x',y') \to (x'',y'')$. Let $\widehat{K}$ be the graph obtained from $K_2$ by performing a 2-sum with $K_1$ along every edge of $K_2$ except $e$; let $e$ remain the distinguished edge of $\widehat{K}$. We claim that $\widehat{K}$ implements the shift $(x,y) \to (x'',y'')$. Since $G \otimes \widehat{K} = (G \otimes K_2) \otimes K_1$, for any $G$, this ought to be true, but we can verify the claim by direct calculation.

Evaluating the rhs of (3), with $K=\widehat{K}$:

$$\begin{aligned}
&\frac{(1-q)T(\widehat{K}\setminus e;x,y)}{T(\widehat{K}\setminus e;x,y)-(x-1)T(\widehat{K}/e;x,y)} \\
&\qquad = \frac{(1-q)T((K_2\setminus e)\otimes K_1;x,y)}{T((K_2\setminus e)\otimes K_1);x,y)-(x-1)T((K_2/e)\otimes K_1;x,y)} \\
&\qquad = \frac{(1-q)T(K_2\setminus e;x',y')}{T(K_2\setminus e;x',y')-(x-1)M(x,y,K_1)T(K_2/e;x',y')} && (7) \\
&\qquad = \frac{(1-q)T(K_2\setminus e;x',y')}{T(K_2\setminus e;x',y')-(x'-1)T(K_2/e;x',y')} && (8) \\
&\qquad = x''. && (9)
\end{aligned}$$

Here, (9) uses (3), and (8) the fact that $(x' - 1) = (x - 1)M(x, y, K_1)$. Equality (7) follows from (5), noting that $K_2/e$ has the same number of edges as $K_2 \setminus e$, but one fewer vertex. A similar calculation holds for $y''$. $\square$

Shifts play a key role in the classical study of the complexity of exact computation of the Tutte polynomial [10], and the same is true in the current investigation. Our keys tools are the following.

**Theorem 2.** *Suppose $(x, y) \in \mathbb{Q}^2$ satisfies $q = (x - 1)(y - 1) \notin \{0, 1, 2\}$. Suppose also that it is possible to shift the point $(x, y)$ to the point $(x', y')$ with $y' \notin [-1, 1]$, and to $(x'', y'')$ with $y'' \in (-1, 1)$. Then there is no FPRAS for* $\text{TUTTE}(x, y)$ *unless* $\text{RP} = \text{NP}$.

**Theorem 3.** *Suppose $(x, y) \in \mathbb{Q}^2$ satisfies $q = (x - 1)(y - 1) \notin \{0, 1, 2\}$. Suppose also that it is possible to shift the point $(x, y)$ to the point $(x', y')$ with $x' \notin [-1, 1]$, and to $(x'', y'')$ with $x'' \in (-1, 1)$. Then there is no FPRAS for* $\text{TUTTE}(x, y)$ *unless* $\text{RP} = \text{NP}$.

Since the notion of "shift" is defined for any class of matroids closed under tensor product, it should be possible to frame statements similar to Theorems 2 and 3 for classes of matroids other than graphic. Although the two theorems are dual to one another in the matroid theoretic sense, they are not equivalent, since the class of graphic matroids is not closed under duality.

The proofs of Theorem 2 and 3 appear in Section 4.

### 3.1 Two halfplanes

**Corollary 4.** *Suppose $(x, y)$ is a point lying in the open half-plane $x < -1$ but not on the hyperbolas $H_0$ or $H_1$. Under the assumption* $\text{RP} \neq \text{NP}$ *there is no FPRAS for* $\text{TUTTE}(x, y)$.

*Proof.* Let $(x, y) \in \mathbb{R}^2$ be a point not on $H_0$ or $H_1$ that satisfies $x < -1$. At the outset, we'll assume further that $(x, y) \notin H_2$ and that $y \neq -1$. There are three cases, depending on $y$. First assume $y > 1$, and observe that $q = (x - 1)(y - 1) < 0$. Using a $k$-stretch, we may shift the point $(x, y)$ to the the point $(x'', y'') = \big(x^k, q/(x^k - 1) + 1\big)$. Now $y'' \in (-1, 1)$ for all sufficiently large even $k$ so Theorem 2 applies. (The trivial shift, taking $(x, y)$ to itself, provides the point $(x', y')$ with $y' \notin [-1, 1]$.) A similar argument, but setting $k$ to be large and odd deals with the situation $y < -1$. Finally, when $y \in (-1, 1)$, a 2-stretch shifts $(x, y)$ to the point $(x', y') = \big(x^2, q/(x^2 - 1) + 1\big) = \big(x^2, (y - 1)/(x + 1) + 1\big)$, with $y' > 1$.

The additional condition $y \neq -1$ may be removed by noting that a 3-stretch shifts $(x, -1)$ to a point $(x'', y'') = \big(x^3, 1 - 2/(x^2 + x + 1)\big)$ with $x'' < -1$ and $y'' \in (-1, +1)$, and we have already seen how to deal with such a point.

Finally, suppose $q = 2$. Like Welsh [23] we will show hardness using an argument of Jerrum and Sinclair [12, (Theorem 14)]. Suppose that $G$ has $n$ vertices and $m$ edges and that $x'$ and $y'$ satisfy $(x'-1)(y'-1) = 2$. Jerrum and Sinclair note that

$$T(G; x', y') = (y'-1)^n (x'-1)^{-\kappa(E)} \sum_{r=0}^{m} N_r (y')^{m-r}$$

where $N_r$ is the number of functions $\sigma : V \to \{-1, 1\}$ with $r$ bichromatic edges. The reader can verify this claim by looking ahead to Equations (10) and (39). Thus, if $G$ has a cut of size $b$ then

$$T(G; x', y') \geq (y'-1)^n (x'-1)^{-\kappa(E)} (y')^{m-b}.$$

Otherwise,

$$T(G; x', y') \leq (y'-1)^n (x'-1)^{-\kappa(E)} 2^n (y')^{m-b+1}.$$

Now consider a point $(x, y)$ on $H_2$ with $x < -1$. Note that $y \in (0, 1)$. Let $k$ be a positive integer with $y^k < 2^{-2n}$ and let $y' = y^k$. Let $x' = 2/(y^k-1)+1$. If we had an FPRAS for $\textsc{Tutte}(x, y)$, we could estimate $T(G; x', y')$ by $k$-thickening. Thus, we could determine whether or not $G$ has a cut of size $b$, giving RP = NP.

□

**Corollary 5.** *Suppose $(x, y)$ is a point lying in the open half-plane $y < -1$ but not on the hyperbolas $H_0$, $H_1$ or $H_2$. Under the assumption* RP ≠ NP *there is no FPRAS for* $\textsc{Tutte}(x, y)$.

*Proof.* Dual to the proof of Corollary 4 (but without the extra argument for $q = 2$). □

Corollaries 4 and 5 exclude the hyperbola $q = 0$. Nevertheless, the arguments of Theorem 2 can be extended to handle the portion of this (degenerate) hyperbola in which $y < -1$. Specifically, in Section 4 we prove the following.

**Lemma 6.** *Suppose $(x, y)$ is a point with $x = 1$ and $y < -1$. Then there is no FPRAS for* $\textsc{Tutte}(x, y)$ *unless* RP = NP.

We do not know whether the arguments of Theorem 3 can be similarly extended to $q = 0$.

The hyperbola $H_2$ is excluded from Theorems 2 and 3 and a separate argument (following Welsh) was used to include $H_2$ within the scope of Corollary 4 which applies to the region $x < -1$. We do not know of a similar argument which applies to $H_2$ in the region $y < -1$ and indeed this hyperbola seems to have a special status in the region $y < -1$, as Lemma 7 shows. Consider the following computational problem.

**Name** #PERFECT MATCHINGS.

**Instance** A graph $G$.

**Output** The number of perfect matchings in $G$.

#PERFECT MATCHINGS is #P-complete, but it is not know whether it has an FPRAS. In Section 4 we prove the following.

**Lemma 7.** *Suppose $(x, y)$ is a point on the hyperbola $H_2$ with $y < -1$. Then* TUTTE$(x, y) \equiv_{\mathrm{AP}}$ #PERFECT MATCHINGS.

**Remark:** For convenience, we allow the graph $G$ in the definition of #PERFECT MATCHINGS to have loops and/or multiple edges. This is without loss of generality, since the perfect matchings of a graph $G$ are in one-to-one correspondence with the perfect matchings of the 3-stretch of $G$ (which has no loops or multiple edges).

## 3.2 The Vicinity of the Origin

In this section, we consider the region given by $|x| < 1$ and $|y| < 1$. We have already seen (in the proof of Corollary 4) that, unless RP = NP, there is no FPRAS for TUTTE$(x, y)$ for any point $(x, y)$ on the hyperbola $(x - 1)(y - 1) = 2$ in this region. The following lemmas give additional regions that do not admit an FPRAS unless RP = NP.

**Lemma 8.** *Suppose $(x, y)$ is a point with $|x| < 1$, $|y| < 1$ and $y < -1 - 2x$ that does not lie on the hyperbola $H_1$. Then there is no FPRAS for* TUTTE$(x, y)$ *unless* RP = NP.

*Proof.* Using Equation (6), a 2-stretch shifts $(x, y)$ to $(x', y')$ with

$$y' = \frac{(x-1)(y-1)}{x^2 - 1} + 1 = \frac{y + x}{x + 1} < -1.$$

Now if $q = (x - 1)(y - 1) \notin \{0, 1, 2\}$, the lemma follows from Theorem 2. As we noted above, the result is already known for $q = 2$. Also, $H_0$ is outside the scope of the lemma. □

**Lemma 9.** *Suppose $(x, y)$ is a point with $|x| < 1$, $|y| < 1$ and $x < -1 - 2y$ that does not satisfy $(x - 1)(y - 1) = 1$. Then there is no FPRAS for* TUTTE$(x, y)$ *unless* RP = NP.

*Proof.* As in the proof of Lemma 8 we can use a 2-thickening together with Theorem 3 to obtain the result when except for $q = 0$, $q = 1$ and $q = 2$. The result is known for $q = 2$ and the cases $q = 0$ and $q = 1$ are excluded from the lemma. □

Lemmas 8 and 9 give two intractable open triangles in the vicinity of the origin. The following lemmas extend intractability to the boundaries. The value 0.29 in the statement of the lemmas has no special significance. We do not know whether the entire boundary is intractable, but the value 0.29 is not best possible — it was chosen because it yields a simple proof.

**Lemma 10.** *Suppose $(x, y)$ is a point with $x = -1$ and $-1 < y < 0.29$, excluding the special point $(x, y) = (-1, 0)$. Then there is no FPRAS for* $\textsc{Tutte}(x, y)$ *unless* RP = NP.

*Proof.* A 2-thickening of $(-1, y)$ gives the point $(x', y') = ((y-1)/(y+1), y^2)$. If $y < 0$ then $x' < -1$ so the result follows from Corollary 4 since $(x', y')$ is not on $H_0$ or $H_1$. Now if $0 < y < 1$ then $x' \in (-1, 0)$ so $|x'| < 1$ and $|y'| < 1$. Now note that if $0 < y < 0.29$ then $y' < -1 - 2x'$ so the result follows from Lemma 8. □

**Lemma 11.** *Suppose $(x, y)$ is a point with $y = -1$ and $-1 < x < 0.29$, excluding the special point $(x, y) = (0, -1)$. Then there is no FPRAS for* $\textsc{Tutte}(x, y)$ *unless* RP = NP.

*Proof.* Dual to the proof of Lemma 10. □

The intractable triangles from Lemma 8 and 9 certainly do not cover all intractable points in the vicinity of the origin. Possibly the whole of the region $|x|, |y| \leq 1$ is intractable (apart from $H_1$ and the special points).

Here is a lemma which adds a little bit to our knowledge in the region. For example, it includes the point $(x, y) = (-0.23, -0.23)$ which has $q > 1.5$ but is not covered by Lemma 8 or 9.

**Lemma 12.** *Suppose $(x, y)$ is a point with $|x| \leq 1$ and $|y| \leq 1$ and $(x - 1)(y - 1) = q > 1.5$ (excluding the special points $(-1, -1)$, $(-1, 0)$ and $(0, -1)$). Then there is no FPRAS for* $\textsc{Tutte}(x, y)$ *unless* RP = NP.

*Proof.* First, note that neither $x$ nor $y$ is 1, since that would make $q = 0$. Also, we don't have $(x, y) = (-1, -1)$ since that is a special point. Suppose $x = -1$. Then $y > -1$. The restriction on $q$ implies $y < 0.25$, so the result follows from Lemma 10. Similarly, the case $y = -1$ follows from Lemma 11. So suppose $|x| < 1$ and $|y| < 1$.

If $q > 2$ then the result follows from Theorem 3. Do a 2-thickening (Equation (6)) to shift to the point

$$(x', y') = \left( \frac{q}{y^2 - 1} + 1, y^2 \right).$$

Note that $y^2 - 1 \in (-1, 0)$ since $|y| < 1$ so $q/(y^2 - 1) < -q < -2$ since $q > 2$. So $x' < -1$. Then apply Theorem 3. The case $q = 2$ is known, as

noted at the beginning of the section. Suppose $3/2 < q < 2$. For a large even integer $k$, do a $k$-thickening to shift $(x, y)$ to the point

$$(x_1, y_1) = \left(\frac{q}{y^k - 1} + 1, y^k\right).$$

Choose $k$ so that $0 < y^k < (2-q)/2$ (this is possible since $q < 2$). Consider $x_1 - 1 = q/(y^k - 1)$. Note that this is in the interval $(-2, -q)$. Now do a 2-stretch to shift $(x_1, y_1)$ to the point $(x', y') = (x_1^2, q/(x_1^2 - 1) + 1)$. Note that

$$y' - 1 = \frac{q}{x_1^2 - 1} < -2$$

where the upper bound of $-2$ follows from the bounds that we derived on $x_1$ and $q > 3/2$. Now use Theorem 2. □

The lemma could certainly be improved. For example, consider the point $(x, y) = (-0.2, 0)$ with $q = 1.2$. An alternating sequence of 14 2-stretches and 2-thickenings shifts this point to a point $(x', y') \sim (-103.1, 0.99)$ so $(x, y)$ had no FPRAS (unless RP = NP) by Theorem 3.

# 4 The reductions

## 4.1 The Multivariate Formulation of the Tutte Polynomial

It is convenient for us to use the multivariate formulation of the Tutte polynomial, also known as the random cluster model [22, 17]. For a graph $G = (V, E)$ with edge weights $w : E \to \mathbb{Q}$ and $q \in \mathbb{Q}$, define the multivariate Tutte polynomial of $G$ to be $Z(G; q, w) = \sum_{A \subseteq E} w(A) q^{\kappa(A)}$, where $w(A) = \prod_{e \in A} w(e)$, and $\kappa(A)$ is the number of connected components in the graph $(V, A)$.

Suppose $(x, y) \in \mathbb{Q}^2$ and $q = (x-1)(y-1)$. For a graph $G = (V, E)$, let $w : E \to \mathbb{Q}$ be the constant function which maps every edge to the value $y - 1$. Then (see, for example [17, (2.26)])

$$T(G; x, y) = (y-1)^{-n}(x-1)^{-\kappa(E)} Z(G; q, w). \tag{10}$$

So $Z$ is a generalisation of $T$ that allows different weights to be assigned independently to different edges. For rationals $q$ and $\gamma$, let $\textsc{MultiTutte}(q, \gamma)$ be the following problem.

**Name** $\textsc{MultiTutte}(q, \gamma)$.

**Instance** A graph $G = (V, E)$ with edge labelling $w$ where $w$ is the constant function mapping every edge to the value $\gamma$.

**Output** $Z(G; q, w)$.

Suppose $(q,\gamma)\in\mathbb{Q}^2$. Equation (10) gives us the reduction

$$\textsc{MultiTutte}(q,\gamma) \le_{\mathrm{AP}} \textsc{Tutte}\left(\tfrac{q}{\gamma}+1,\gamma+1\right).$$

If $(x,y)\in\mathbb{Q}^2$ and neither $x$ nor $y$ is 1 then Equation (10) gives us the corresponding reduction

$$\textsc{Tutte}(x,y) \le_{\mathrm{AP}} \textsc{MultiTutte}((x-1)(y-1),y-1).$$

Not surprisingly, the notion of a shift from §3 may be re-expressed in terms of the new parameters. Doing so has the advantage of allowing us to apply shifts to individual edges of a graph, as opposed to the whole graph. This idea is explored in [17, §4.6]. We derive the equations that we need here in order to fix the notation and explore the concepts. As in §3, let $K$ be a graph with distinguished edge $e$, and suppose that $K$ has constant edge weight $\alpha\in\mathbb{Q}$. Define

$$\alpha' = \frac{q\,Z(K/e;q,\alpha)-Z(K\setminus e;q,\alpha)}{Z(K\setminus e;q,\alpha)-Z(K/e;q,\alpha)} \tag{11}$$

and

$$N(q,\alpha,K) = \frac{q(q-1)}{Z(K\setminus e;q,\alpha)-Z(K/e;q,\alpha)}.$$

Let $G=(V,E)$ be a graph, $f\in E$ and $w':E\to\mathbb{Q}$ a weighting such that $w'(f)=\alpha'$. Denote by $G_f$ the 2-sum of $G$ and $K$ along $f$. Let $w$ be the weight function on $G_f$ that gives every edge of $K$ weight $\alpha$ and inherits the remaining weights from $w'$. We will show below that

$$Z(G;q,w') = N(q,\alpha,K)\,Z(G_f;q,w). \tag{12}$$

One way to capture (12) informally is to say that a single edge of weight $\alpha'$ may be simulated by a subgraph $K$ whose edges have weight $\alpha$.

Suppose that the denominator of (11) is non-zero. In this case, the point $(q,\alpha')$ is well-defined and we say that $(q,\alpha)$ is *shifted to* the point $(q,\alpha')$ by $K$. In this case, $N(q,\alpha,K)$ is also well-defined, so Equation (12) gives us an efficient algorithm for approximating $Z(G;q,w')$ by using an subroutine for computing $Z(G_f;q,w)$.

For the derivation of (11) and (12), let $v$ and $v'$ be the endpoints of $f$. Let $S$ be the set of subsets of $E-\{f\}$ which connect $v$ and $v'$ and let $T$ be the set of all other subsets of $E-\{f\}$. Then $Z(G;q,w')=Z_S+Z_T$, where

$$Z_S = \sum_{A'\in S} w(A')q^{k(A')}(1+\alpha'),$$

and

$$Z_T = \sum_{A'\in T} w(A')q^{k(A')}\left(1+\tfrac{\alpha'}{q}\right).$$

Similarly, $Z(G_f, q, w) = Z_{f,S} + Z_{f,T}$, where

$$Z_{f,S} = \sum_{A' \in S} w(A') q^{k(A')} \frac{Z(K/e; q, \alpha)}{q},$$

and

$$Z_{f,T} = \sum_{A' \in T} w(A') q^{k(A')} \frac{Z(K \setminus e; q, \alpha)}{q^2}.$$

Now the equation for $\alpha'$ comes from the following argument. Suppose we could define $\alpha'$ such that

$$\frac{Z_S}{Z_T} = \frac{Z_{f,S}}{Z_{f,T}}, \tag{13}$$

and $N(q, \alpha, K) = Z_S / Z_{f,S}$. Then (12) would hold as desired. Now note that (11) entails (13).

The shifts that we have defined here are consistent with the usage in §3. In particular, suppose that $(x, y)$ is shifted to the point $(x', y')$ by a graph $K$ with distinguished edge $e$. As long as $e$ is not a bridge of $K$ then taking $\alpha = y - 1$ and $\alpha' = y' - 1$ and $q = (x-1)(y-1)$ we find (from Equations (4) and (10) and (11)) that the same graph $K$ shifts $(q, \alpha)$ to $(q, \alpha')$.

Thus, the equation describing stretching and thickening, Equation (6), can be translated as follows. (See, for example, [17, (4.20), (4.26)])

$$\begin{aligned} \frac{q}{\alpha'} &= \left(\frac{q}{\alpha} + 1\right)^k - 1, \qquad \text{for a } k\text{-stretch;} \\ \alpha' &= (\alpha + 1)^k - 1, \qquad \text{for a } k\text{-thickening.} \end{aligned} \tag{14}$$

We now generalise the computational problem $\textsc{MultiTutte}(q, \gamma)$ defined earlier. For rationals $q, \alpha_1, \dots, \alpha_k$, $\textsc{MultiTutte}(q; \alpha_1, \dots, \alpha_k)$ is the problem:

**Name** $\textsc{MultiTutte}(q; \alpha_1, \dots, \alpha_k)$.

**Instance** A graph $G = (V, E)$ with edge labelling $w : E \to \{\alpha_1, \dots, \alpha_k\}$.

**Output** $Z(G; q, w)$.

### 4.2 Proof of Theorem 2

The decision problem MINIMUM 3-WAY CUT is:

**Instance** A simple graph $G = (V, E)$ with three distinguished vertices ("terminals") $t_1, t_2, t_3 \in V$, and an integer bound $b$.

**Output** Is there a set of at most $b$ edges whose removal from $G$ disconnects $t_i$ from $t_j$ for every $i, j \in \{1, 2, 3\}$ with $i \neq j$?

It was shown to be NP-complete by Dahlhaus et al. [3].

**Lemma 13.** *Suppose $q \in \mathbb{Q} \setminus \{0,1,2\}$, and that $\alpha_1, \alpha_2 \in \mathbb{Q}$ satisfy $\alpha_1 \notin [-2,0]$ and $\alpha_2 \in (-2,0)$. Then there is no FPRAS for* $\textsc{MultiTutte}(q;\alpha_1,\alpha_2)$ *unless* RP = NP.

*Proof.* Suppose $G = (V, E, t_1, t_2, t_3)$ is an arbitrary instance of Min 3-way Cut. Without loss of generality assume $G$ is connected, and for convenience let $n = |V|$ and $m = |E|$. Our ultimate goal is to construct an instance $(G', w')$ of $\textsc{MultiTutte}(q;\alpha_1,\alpha_2)$ such that $Z(G'; q, w')$ is a close approximation to the number of minimum 3-way cuts in $G$. The size of a minimum cut will be a by-product of the of the reduction.

As an intermediate goal, we'll construct a weighted graph $(\widehat{G} = (\widehat{V}, \widehat{E}), w)$ such that $Z(\widehat{G}; q, w)$ is a close approximation to the number of minimum 3-way cuts in $G$ where $w : \widehat{E} \to \{\beta_1, \beta_2\}$ for some conveniently-chosen values $\beta_1$ and $\beta_2$. The final step of the proof will be to relate these convenient values to the specified ones, namely $\alpha_1$ and $\alpha_2$. We will require $\beta_1$ to be sufficiently large, in particular, let $\overline{q} = \max(|q|, 1)$. Let $M = 8 \times 2^m \overline{q}^n$. We will require

$$\beta_1 \geq M. \tag{15}$$

We will also require $\beta_2$ to be sufficiently close to $-1$. In particular, we will choose a small value $\delta$ (see Equations (21) and (22), depending on $m$, $q$ and $n$. We will require $|1 + \beta_2| \leq \delta$.

The construction of $(\widehat{G} = (\widehat{V}, \widehat{E}), w)$ is very direct: $\widehat{V} = V$, $\widehat{E} = E \cup T$, where $T = \big\{\{t_1, t_2\}, \{t_2, t_3\}, \{t_1, t_3\}\big\}$, and

$$w(e) = \begin{cases} \beta_1, & \text{if } e \in E; \\ \beta_2, & \text{otherwise.} \end{cases}$$

Now, letting

$$\mathcal{A}_{1|2,3} = \{A \subseteq E : t_1 \not\sim_A t_2 \text{ and } t_1 \not\sim_A t_3 \text{ and } t_2 \sim_A t_3\},$$

etc, where $\sim_A$ denotes the binary relation "is connected to" in the graph $(V, A)$, we may express the multivariate Tutte polynomial of $\widehat{G}$ as

$$Z(\widehat{G}; q, w) = \varSigma_{1|2|3} + \varSigma_{1|2,3} + \varSigma_{2|1,3} + \varSigma_{3|1,2} + \varSigma_{1,2,3}, \tag{16}$$

where, e.g.,

$$\varSigma_{1|2,3} = \sum_{A \in \mathcal{A}_{1|2,3}} \sum_{B \subseteq T} w(A \cup B) q^{\kappa(A \cup B)}.$$

The overview of the proof is as follows: we show that for $\beta_1$ sufficiently large and $\beta_2$ sufficiently close to $-1$, the last four terms on the r.h.s. of (16) are negligible in comparison with the first, and that the first term, $\varSigma_{1|2|3}$,

counts minimum 3-way cuts in $G$ (approximately, and up to an easily computable factor). Up to symmetry there are three essentially distinct terms in (16), and we consider them in turn. First,

$$\begin{aligned}
|\Sigma_{1,2,3}| &= \left| \sum_{A\in\mathcal{A}_{1,2,3}} \sum_{B\subseteq T} w(A\cup B) q^{\kappa(A\cup B)} \right| \\
&= \left| \sum_{A\in\mathcal{A}_{1,2,3}} w(A) q^{\kappa(A)} (1+\beta_2)^3 \right| \\
&\le (2\beta_1)^m |q|\, \bar{q}^{\,n-1} \delta^3, \qquad (17)
\end{aligned}$$

Here we have used $1 \le \kappa(A) \le n$. Next,

$$\begin{aligned}
|\Sigma_{1|2,3}| &= \left| \sum_{A\in\mathcal{A}_{1|2,3}} \sum_{B\subseteq T} w(A\cup B) q^{\kappa(A\cup B)} \right| \\
&= \left| \sum_{A\in\mathcal{A}_{1|2,3}} w(A) q^{\kappa(A)-1} (q + 2\beta_2 + \beta_2^2)(1+\beta_2) \right| \\
&\le (2\beta_1)^m |q|\, \bar{q}^{\,n-2} \,(\overline{q} + 4 + 4)\delta \\
&\le 9(2\beta_1)^m |q|\, \bar{q}^{\,n-1} \delta. \qquad (18)
\end{aligned}$$

Here we used $|\beta_2| \le 2$ and $2 \le \kappa(A) \le n$. Last,

$$\begin{aligned}
\Sigma_{1|2|3} &= \sum_{A\in\mathcal{A}_{1|2|3}} \sum_{B\subseteq T} w(A\cup B) q^{\kappa(A\cup B)} \\
&= \sum_{A\in\mathcal{A}_{1|2|3}} w(A) q^{\kappa(A)-2} \big(q^2 + 3\beta_2 q + 3\beta_2^2 + \beta_2^3\big) \\
&= C(\beta_2) \sum_{A\in\mathcal{A}_{1|2|3}} w(A) q^{\kappa(A)-2}, \qquad (19)
\end{aligned}$$

where

$$C(\beta_2) = (q-1)(q-2) + 3(q-1)(1+\beta_2) + (1+\beta_2)^3.$$

Note that

$$\big|C(\beta_2) - (q-1)(q-2)\big| \le 3\,|q-1|\,\delta + \delta^3,$$

The crucial fact is that $C(\beta_2)$ remains bounded away from 0 as $\delta \to 0$ (and hence $\beta_2 \to -1$), provided (as we are assuming) $q \notin \{1,2\}$, whereas expressions (17) and (18) tend to 0 as $\delta \to 0$.

Now denote by $\mathcal{A}^{(i)}_{1|2|3}$ the set of all subsets in $\mathcal{A}_{1|2|3}$ of size $i$. Let $c$ be the size of a minimum 3-way cut in $G$, and $N$ be the number of such cuts. Then

$$
\begin{aligned}
\frac{\Sigma_{1|2|3}}{C(\beta_2)} &= \sum_{A\in\mathcal{A}^{(m-c)}_{1|2|3}} w(A)q^{\kappa(A)-2} + \sum_{i=0}^{m-c-1}\sum_{A\in\mathcal{A}^{(i)}_{1|2|3}} w(A)q^{\kappa(A)-2} \\
&= N\beta_1^{m-c}q + \sum_{i=0}^{m-c-1}\sum_{A\in\mathcal{A}^{(i)}_{1|2|3}} w(A)q^{\kappa(A)-2}.
\end{aligned}
$$

Then

$$\frac{\Sigma_{1|2|3}}{C(\beta_2)\beta_1^{m-c}q} - N$$

is equal to

$$\left(\sum_{i=0}^{m-c-1}\sum_{A\in\mathcal{A}^{(i)}_{1|2|3}} w(A)q^{\kappa(A)-2}\right)\bigg/\left(\beta_1^{m-c}q\right),$$

so crudely upper-bounding the absolute value of the right-hand-side, we get

$$\left|\frac{\Sigma_{1|2|3}}{C(\beta_2)\beta_1^{m-c}q} - N\right| \leq \frac{2^m\bar{q}^n}{\beta_1}. \tag{20}$$

Now set $\delta$ to satisfy both

$$|C(\beta_2)| \geq \tfrac{1}{2}|C(-1)| \tag{21}$$

and

$$\delta \leq \frac{|C(-1)|}{448(2\beta_1)^m\bar{q}^n}. \tag{22}$$

Now (15) ensures that the r.h.s. of (20) is at most $\frac{1}{8}$, while (17), (18), (19), (21) and (22) ensure

$$\left|\frac{\Sigma_{1|2|3} - Z(\widehat{G};q,w)}{C(\beta_2)\beta_1^{m-c}q}\right| = \left|\frac{\Sigma_{1|2,3}+\Sigma_{2|1,3}+\Sigma_{3|1,2}+\Sigma_{1,2,3}}{C(\beta_2)\beta_1^{m-c}q}\right| \leq \frac{1}{8}.$$

Combining this inequality with (20), the bottom line is

$$\left|\frac{Z(\widehat{G};q,w)}{C(\beta_2)\beta_1^{m-c}q} - N\right| \leq \frac{1}{4}. \tag{23}$$

If we knew $Z(\widehat{G};q,w)$, we could determine $c$ — it is the unique integer such that (23) provides an estimate for $N$ that lies in the range $[1, 2^m]$. The value of $c$ is unique since $\beta_1 \geq M > 2^m$.)

In fact, we do not need an exact value of $Z(\widehat{G};q,w)$ — an approximate value will do. In particular, an FPRAS for $Z(\widehat{G};q,w)$ would give a randomised polynomial-time algorithm for computing $c$, which would show

RP = NP. For details about approximation accuracy, see [4], especially the final three paragraphs of the proof of Theorem 3.

Finally we need to relate our weights $\beta_1, \beta_2$ to the given ones $\alpha_1, \alpha_2$. Let positive integers $k_1, k_2$ satisfy $(\alpha_1 + 1)^{k_1} - 1 \geq M$ and $\left|(\alpha_2 + 1)^{k_2}\right| < \delta$. Let $K_1$ be a 2-vertex graph with $k_1 + 1$ parallel edges, each of weight $\alpha_1$. Recall that taking a 2-sum with $K_1$ implements a $k_1$ thickening. Let $K_2$ be a 2-vertex graph with $k_2 + 1$ parallel edges, each of weight $\alpha_2$. Let $G'$ be the graph derived from $\widehat{G}$ by taking the 2-sum of each weight $\beta_1$ edge with $K_1$ and taking the 2-sum of each weight $\beta_2$ edge with $K_2$. Call the resulting graph $G'$ and its weighting $w'$. By repeated application of (12), $Z(\widehat{G}; q, w) = N(q, \alpha_1, K_1)^m N(q, \alpha_2, K_2)^3 Z(G'; q, w')$. By setting $\beta_i = (\alpha_i + 1)^{k_i} - 1$, for $i = 1, 2$, we satisfy $\beta_1 > M$ and $|\beta_2 + 1| \leq \delta$, as required by our reduction. (This is by (14) and the definitions of $k_1$ and $k_2$.) Finally observe that $k_1 = O(m)$ and $k_2 = O(m^2)$, so the size of $G'$ is polynomially bounded.

Thus an FPRAS for MultiTutte$(q; \alpha_1, \alpha_2)$ would yield a polynomial-time randomised algorithm for computing the size of a minimum 3-way cut, which would entail RP = NP. □

Using Lemma 13, we can now prove Theorem 2.

**Theorem 2** *Suppose $(x, y) \in \mathbb{Q}^2$ satisfies $q = (x-1)(y-1) \notin \{0, 1, 2\}$. Suppose also that it is possible to shift the point $(x, y)$ to the point $(x', y')$ with $y' \notin [-1, 1]$, and to $(x'', y'')$ with $y'' \in (-1, 1)$. Then there is no FPRAS for* Tutte$(x, y)$ *unless* RP = NP.

*Proof.* Let $\alpha = y - 1$ and $\alpha_1 = y' - 1$ and $\alpha_2 = y'' - 1$. Note that $\alpha_1 \notin [-2, 0]$ and $\alpha_2 \in (-2, 0)$. Let $(K', e')$ be a graph that shifts $(x, y)$ to $(x', y')$ and note that $(K', e')$ also shifts $(q, \alpha)$ to $(q, \alpha_1)$. Similarly, suppose $(K'', e'')$ shifts $(x, y)$ to $(x'', y'')$ and therefore shifts $(q, \alpha)$ to $(q, \alpha_2)$.

Suppose $(G, w)$ is an instance of MultiTutte$(q; \alpha_1, \alpha_2)$ with $m_1$ edges with weight $\alpha_1$ and $m_2$ edges with weight $\alpha_2$. Denote by $\widehat{G}$ the graph derived from $G$ by taking a 2-sum with $(K', e')$ along every edge with weight $\alpha_1$ and taking a 2-sum with $(K'', e'')$ along every edge with weight $\alpha_2$. Let $\hat{w}$ be the constant weight function which assigns weight $\alpha$ to every edge in $\widehat{G}$.

Then by repeated use of Equation (12),

$$Z(G; q, w) = N(q, \alpha, K_1)^{m_1} N(q, \alpha, K_2)^{m_2} Z(\widehat{G}; q, \hat{w}).$$

Thus by Equation (10),

$$Z(G; q, w) = N(q, \alpha, K_1)^{m_1} N(q, \alpha, K_2)^{m_2} (y-1)^n (x-1)^\kappa T(\widehat{G}; x, y),$$

where $n$ is the number of vertices in $\widehat{G}$, and $\kappa$ is the number of connected components in $\widehat{G}$.

Thus an FPRAS for Tutte$(x, y)$ would yield an FPRAS for the problem MultiTutte$(q; \alpha_1, \alpha_2)$, contrary to Lemma 13. □

### 4.3 Extending to $q = 0$

Formally, the multivariate Tutte polynomial $Z(G; q, w) = \sum_{A \subseteq E} w(A) q^{\kappa(A)}$ is not very interesting at $q = 0$ because $\kappa(A) \geq 1$ so $Z(G; q, w) = 0$. Sokal [17] treats the $q = 0$ case as a limit, but for the purpose of approximation complexity it is more convenient to work with the polynomial $Z(G; q, w) q^{-\kappa(E)}$. We will focus on the case in which $G$ is connected, so we define $R(G; q, w) = Z(G; q, w) q^{-1}$. Note that

$$R(G; 0, w) = \sum_{A \subseteq E : \kappa(A) = 1} w(A). \tag{24}$$

This is the reliability polynomial, and corresponds to the $x = 1$ component of the hyperbola $H_0$.

We can express shifts in terms of $R(G; q, w)$. For example, Equation (14) does not tell us anything useful about stretching for $q = 0$ (due to cancellation) but the same reasoning that we used to derive (11) and (12) gives us the following version of these equations for the case in which $G_f$ is a $k$-stretch (specifically, $G_f$ is the 2-sum of $G$ and a cycle on $k + 1$ vertices along edge $f$):

$$\alpha' = \frac{\alpha}{k} \tag{25}$$

and

$$R(G; 0, w') = \frac{1}{k \alpha^{k-1}} R(G_f; 0, w). \tag{26}$$

As in the general case, we assume $w'$ is a weight function on $G$ with $w(f) = \alpha'$ and that $w$ inherits its weights from $w'$ except that the new edges in the stretch are given weight $\alpha$. The derivation of (25) and (26) follows the derivation of (11) and (12). Specifically, let $S$ (respectively, $T$) be the set of all subsets $A' \subseteq E - \{f\}$ with $\kappa(A') = 1$ (respectively, $\kappa(A') = 2$ and $\kappa(A' \cup \{f\}) = 1$). Then $R(G; 0, w') = R_S + R_T$ where

$$R_S = \sum_{A' \subseteq S} w(A')(1 + \alpha'), \tag{27}$$

$$R_T = \sum_{A' \subseteq T} w(A') \alpha'; \tag{28}$$

and $R(G_f; 0, w') = R_{f,S} + R_{f,T}$, where

$$R_{f,S} = \sum_{A' \subseteq S} w(A')(\alpha^k + k\alpha^{k-1}), \tag{29}$$

$$R_{f,T} = \sum_{A' \subseteq T} w(A') \alpha^k. \tag{30}$$

Similarly, for the case in which $G_f$ is a $k$-thickening, we get

$$R(G; 0, w') = R(G_f; 0, w), \tag{31}$$

with $\alpha'$ as in Equation (14)

Now let ZEROMULTITUTTE$(\alpha_1, \ldots, \alpha_k)$ be the following problem.

**Name** ZEROMULTITUTTE$(\alpha_1, \ldots, \alpha_k)$.

**Instance** A connected graph $G = (V, E)$ with edge labelling $w : E \to \{\alpha_1, \ldots, \alpha_k\}$.

**Output** $R(G; 0, w)$.

An examination of the proof of Lemma 13 gives the following lemma.

**Lemma 14.** *Suppose that $\alpha_1, \alpha_2 \in \mathbb{Q}$ satisfy $\alpha_1 \notin [-2, 0]$ and $\alpha_2 \in (-2, 0)$. Then there is no FPRAS for* ZEROMULTITUTTE$(\alpha_1, \alpha_2)$ *unless* RP = NP.

The proof of Lemma 14 follows that of Lemma 13. By analogy to Equation (16) we may express $R(\widehat{G}; 0, w)$ as a sum of terms of the form $\Sigma_{1|2|3}$. Then

$$\Sigma_{1|2|3} = \sum_{A \in \mathcal{A}_{1|2|3} : \kappa(A) = 3} w(A)(\beta_2^3 + 3\beta_2^2),$$

and the other terms all have factors of $\delta$. By analogy to Equation (20) we get

$$\left| \frac{\Sigma_{1|2|3}}{(\beta_2^3 + 3\beta_2^2)\beta_1^{m-c}} - N \right| \leq \frac{2^m}{\beta_1}. \tag{32}$$

Using Lemma 14, we can now prove Lemma 6.

**Lemma 6.** *Suppose $(x, y)$ is a point with $x = 1$ and $y < -1$. Then there is no FPRAS for* TUTTE$(x, y)$ *unless* RP = NP.

*Proof.* Let $(x, y)$ be a point with $x = 1$ and $y < -1$. Let $\alpha = y - 1$ and $q = 0$. Note that $\alpha \notin [-2, 0]$. Let $k = \lfloor -\alpha \rfloor$ and let $\alpha_2 = \alpha/k$. Note that $\alpha_2 \in (-2, 0)$, and by Equation (26), a $k$-stretch shifts $(q = 0, \alpha)$ to $(q = 0, \alpha_2)$. Suppose $(G, w)$ is an instance of ZEROMULTITUTTE$(\alpha, \alpha_2)$ with $m_2$ edges with weight $\alpha_2$. Denote by $\widehat{G}$ the graph derived from $G$ by applying a $k$-stretch to each of these $m_2$ edges. Let $\hat{w}$ be the constant weight function which assigns weight $\alpha$ to every edge in $\widehat{G}$. Then by repeated use of Equation (26),

$$R(G; 0, w) = \left( \frac{1}{k\alpha^{k-1}} \right)^{m_2} R(\widehat{G}; 0, \hat{w}).$$

Using Equation (24),

$$R(G; 0, w) = \left( \frac{1}{k\alpha^{k-1}} \right)^{m_2} \sum_{A \subseteq E : \kappa(A) = 1} (y - 1)^{|A|},$$

where $E$ is the edge set of $\widehat{G}$, which is connected since $G$ is. Thus, by the definition of the Tutte polynomial (1),

$$R(G;0,w) = \left(\frac{1}{k\alpha^{k-1}}\right)^{m_2} (y-1)^{n-1} T(\widehat{G};x,y),$$

where $n$ is the number of vertices of $\widehat{G}$. So an FPRAS for $\textsc{Tutte}(x,y)$ would enable us to approximate $R(G;0,w)$, contrary to Lemma 14.

□

### 4.4 Proof of Theorem 3

The following is dual to Lemma 13.

**Lemma 15.** *Suppose $q \in \mathbb{Q} \setminus \{0,1,2\}$, and that $\alpha_1, \alpha_2 \in \mathbb{Q} - \{0\}$ satisfy $q/\alpha_1 \notin [-2,0]$ and $q/\alpha_2 \in (-2,0)$. Then there is no FPRAS for* $\textsc{MultiTutte}(q;\alpha_1,\alpha_2)$ *unless* RP = NP.

*Proof.* We reuse the construction that Frederickson and Ja' Ja' designed in order to prove that CONNECTED BRIDGE-CONNECTIVITY AUGMENTATION (CBRA) is NP-complete [8, Thm 2], though we'll change the edge weights to suit our purpose. For convenience the construction will be repeated here. We start with an instance of the 3-d Matching Problem: $W$, $X$ and $Y$ are disjoint $n$-element sets, and $M \subseteq W \times X \times Y$ a set of triples. We want to know how many "3-d matchings" there are in $M$. A *3-d matching* is a subset $M' \subseteq M$ of $n$ triples such that every element of $W \cup X \cup Y$ is included in some triple in $M'$. For convenience, we'll enumerate the elements of the ground set $W = \{w_1, \dots, w_n\}$, $X = \{x_1, \dots, x_n\}$, and $Y = \{y_1, \dots, y_n\}$.[2]

Our ultimate goal is to construct an instance $(G', w')$ of $\textsc{MultiTutte}(q;\alpha_1,\alpha_2)$ such that $Z(G';q,w')$ is determined, to a high degree of accuracy, by the number of solutions to the instance of #3-D MATCHING. In particular, using an estimate of $Z(G';q,w')$, we'll be able to decide, with high probability, whether the number of solutions to the matching instance is zero or strictly positive. As an intermediate goal, just as in the proof of Lemma 13, we'll construct a weighted graph $(G = (V,E), w)$ that has the desired properties, as described above, except that $w : V \to \{\beta_1, \beta_2\}$, where $\beta_1$ and $\beta_2$ are set to convenient non-zero values. The final step of the proof will be to relate these convenient values to the specified ones, namely $\alpha_1$ and $\alpha_2$. The requirements on $\beta_1$ and $\beta_2$ are similar to the ones that we used in the proof of Lemma 13. In particular, we will require, for a small $\varepsilon \le 1$, that $|\beta_1/q| \le \varepsilon$ (so the absolute value of $q/\beta_1$ is big). We will also require for a small $\delta \le \frac{1}{2}$ that $|1 + q/\beta_2| \le \delta$ (so $\beta_2$ is close to $-q$). We will require $\varepsilon$ and $\delta$ to be sufficiently small — the exact requirements will be given later.

[2] We'll stick, as far as possible, to the notation of [8], though occasional changes are needed to avoid clashes.

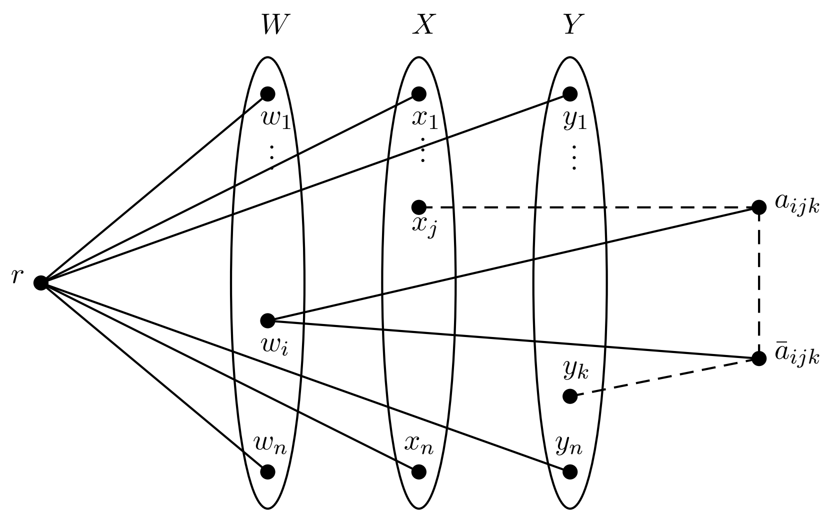


Figure 2: The construction of the graph $G$ in the proof of Lemma 15. The edges relating to just one generic triple $(w_i, x_j, y_k) \in M$ are shown. "Link edges" are dashed.

The vertex set of $G$ (refer to Figure 2) is

$$V = \{r\} \cup W \cup X \cup Y \cup \big\{a_{ijk}, \bar{a}_{ijk} : (w_i, x_j, y_k) \in M\big\},$$

and the edge set $E = T \cup L$ where

$$\begin{aligned} T = \big\{&\{r, w_i\}, \{r, x_i\}, \{r, y_i\} : 1 \le i \le n\big\} \\ &\cup \big\{\{w_i, a_{ijk}\}, \{w_i, \bar{a}_{ijk}\} : (w_i, x_j, y_k) \in M\big\} \end{aligned}$$

is the set of "tree edges" and

$$L = \big\{\{x_j, a_{ijk}\}, \{a_{ijk}, \bar{a}_{ijk}\}, \{\bar{a}_{ijk}, y_k\} : (w_i, x_j, y_k) \in M\big\}.$$

the "link edges". Observe that $(V, T)$ is a tree, and that edges in $L$ join leaves in the tree. For $e \in E$, assign weight $w(e) = \beta_2$ if $e \in T$ is a tree edge, and $w(e) = \beta_1$ if $e \in L$ is a link edge.

We're interested in evaluating $Z(G; q, w)$:

$$\begin{aligned} Z(G; q, w) &= \sum_{A \subseteq E} w(A) q^{\kappa(A)} \\ &= \sum_{B \subseteq L} \sum_{C \subseteq T} w(B \cup C) q^{\kappa(B \cup C)} \\ &= \sum_{B \subseteq L} h(B, \beta_2)\, \beta_1^{|B|}, \end{aligned} \tag{33}$$

where

$$h(B; \beta_2) = \sum_{C \subseteq T} \beta_2^{|C|} q^{\kappa(B \cup C)} = Z(G \setminus \overline{B}/B; q, w) \tag{34}$$

Here, $G \setminus \overline{B}/B$ denotes the graph obtained from $G$ by deleting edges in $\overline{B} = L \setminus B$ and contracting edges in $B$.

Let $m = |M|$, and note that $|V| = 3n + 2m + 1$, $|T| = 3n + 2m$ and $|L| = 3m$. Our calculation of $Z(G; q, w)$ is greatly simplified if we take $\beta_2$ to be exactly $-q$, rather than merely a close approximation. So let's first determine, as a function of $\delta$, the absolute error we would introduce by replacing $\beta_2$ by $-q$. Denote by $\widetilde{w} : E \to \mathbb{Q}$ the weight function

$$\widetilde{w}(e) = \begin{cases} -q, & \text{if } e \in T; \\ w(e) = \beta_1, & \text{otherwise.} \end{cases}$$

We wish to estimate the absolute error $\bigl|Z(G; q, w) - Z(G; q, \widetilde{w})\bigr|$. Set $\bar{q} = \max\{|q|, 1\}$; then either $|q|/|\beta_2| \geq 1$, in which case $|\beta_2| \leq \bar{q}$ or $|q|/|\beta_2| < 1$. In this case, since $|1 + q/\beta_2| \leq 1/2$, $|1 + q/\beta_2| = 1 - |q|/|\beta_2| \leq 1/2$, so $|\beta_2| \leq 2|q|$. We conclude that, in either case, $|q|, |\beta_2| \leq 2\bar{q}$. Furthermore, for all $i \geq 1$, we have

$$\beta_2^i - (-q)^i = (\beta_2 + q) \sum_{j=0}^{i-1} \beta_2^j (-q)^{i-1-j} \leq i(2\bar{q})^{i-1} 2\bar{q}\delta = i(2\bar{q})^i \delta,$$

since $|\beta_2 + q| \leq |\beta_2|\,|1 + q/\beta_2| \leq 2\bar{q}\delta$. Expanding $h(B, \beta_2)$ and $h(B, -q)$ according to (34), and comparing term-by-term, we find that

$$\begin{aligned} |h(B, \beta_2) - h(B, -q)| &\leq 2^{|T|} |T| \,(2\bar{q})^{|T|} \delta\, \bar{q}^{|V|} \\ &\leq |T| (2\bar{q})^{|V| + |T|} \delta \\ &= (3n + 2m)(2\bar{q})^{6n + 4m + 1} \delta. \end{aligned}$$

So from (33), recalling $|\beta_1| \leq |q|\varepsilon \leq \bar{q}$,

$$\begin{aligned} \bigl|Z(G; q, w) - Z(G; q, \widetilde{w})\bigr| &\leq 2^{|L|} \bar{q}^{|L|} (3n + 2m)(2\bar{q})^{6n + 4m + 1} \delta \\ &\leq (3n + 2m)(2\bar{q})^{6n + 7m + 1} \delta. \end{aligned} \tag{35}$$

We'll chose $\delta$ later to make this estimate small enough.

We now proceed with our calculation, using $\widetilde{w}$ in place of $w$, i.e., $-q$ in place of $\beta_2$. Partition sum (33) in two pieces:

$$Z(G; q, \widetilde{w}) = \varSigma_{\leq} + \varSigma_{>},$$

where

$$\varSigma_{\leq} = \sum_{B \subseteq L : |B| \leq n + m} h(B, -q)\, \beta_1^{|B|} \quad \text{and} \quad \varSigma_{>} = \sum_{B \subseteq L : |B| > n + m} h(B, -q)\, \beta_1^{|B|}.$$

Set $Q = (-1)^n q^{2n + m + 1} (q - 1)^m (q - 2)^n$, and note that $Q \neq 0$. We'll show:

1. If $|B| < n+m$ then $h(B,-q) = 0$.
2. If $|B| = n+m$ then[3]

$$h(B,-q) = \begin{cases} Q, & \text{if } (V, T \cup B) \text{ is bridge connected;} \\ 0, & \text{otherwise.} \end{cases}$$

3. The set $\{B : |B| = n+m \text{ and } (V, T \cup B) \text{ is bridge connected}\}$ is in 1-1 correspondence with the set of solutions to the instance of #3-D Matching.

Observations 1–3 entail

$$\Sigma_{\le} = QN{\beta_1}^{n+m},$$

where $N$ is the number of solutions to the #3-D Matching instance. On the other hand, $\Sigma_{>}$ is crudely bounded as follows:

$$\begin{aligned} |\Sigma_{>}| &= \sum_{B \subseteq L : |B| > n+m} h(B,-q) q^{|B|} \left(\frac{\beta_1}{q}\right)^{|B|} \\ &\le \sum_{B \subseteq L} \left| h(B,-q) \bar{q}^{|B|} (\beta_1/q)^{n+m+1} \right| \\ &\le 2^{|L|} 2^{|T|} \bar{q}^{\,|T|} \bar{q}^{\,|V|}\, \bar{q}^{|L|} |\beta_1/q|^{n+m+1} \\ &\le (2\bar{q})^{6n+7m+1} |\beta_1/q|^{n+m+1}. \end{aligned}$$

Let $\widehat{Q} = q^{n+m}Q$. Now, setting $\varepsilon$ (the bound on $|\beta_1/q|$) so that

$$(2\bar{q})^{6n+7m+1}\varepsilon \le \tfrac{1}{8}|\widehat{Q}|,$$

we have

$$\left| \frac{Z(G;q,\widetilde{w})}{Q\beta_1^{n+m}} - N \right| = \left| \frac{\Sigma_{\le} + \Sigma_{>}}{Q\beta_1^{n+m}} - N \right| = \left| \frac{\Sigma_{>}}{Q\beta_1^{n+m}} \right| = \left| \frac{\Sigma_{>}}{\widehat{Q}(\beta_1/q)^{n+m}} \right| \le \frac{1}{8}. \tag{36}$$

Then, according to (35), setting

$$(3n+2m)(2\bar{q})^{6n+7m+1}\delta \le \tfrac{1}{8} Q\beta_1^{n+m},$$

ensures

$$\left| \frac{Z(G;q,w) - Z(G;q,\widetilde{w})}{Q\beta_1^{n+m}} \right| \le \frac{1}{8}. \tag{37}$$

[3] *Bridge connected* is a synonym for 2-edge-connected, i.e., connected and having no bridge, which is an edge whose removal would disconnect the graph.

Combining (36) and (37) yields the required estimate

$$\left|\frac{Z(G;q,w)}{Q\beta_1^{n+m}} - N\right| \leq \frac{1}{4}.$$

It remains to verify the three observations. Suppose the graph $H = G \setminus \overline{B}/B$ contains a bridge $e$. Then

$$Z(H;q,\widetilde{w}) = \sum_{A' \subseteq E(H)-\{e\}} \widetilde{w}(A')q^{\kappa(A')}\left(\frac{-q}{q}+1\right) = 0,$$

where the $-q/q$ comes from including $e$ in $A$, which gives a weight of $-q$ but reduces the number of components by one and the 1 comes from excluding $e$ from $A$. The tree $(V,T)$ has $2(n+m)$ leaves, so if $|B| < n+m$ there are at least two vertices in $(V, T \cup B)$ of degree one. The unique edge $e$ incident at either of these vertices is a bridge, and is a member of $T$; it is clearly also a bridge in $H$. This deals with Observation 1.

Suppose $B \subseteq L$ is a set of link-edges of size $n+m$ such that $(V, T \cup B)$ is bridge connected. Every leaf of $(V,T)$ must have some edge of $B$ incident at it, and hence exactly one. Call such a $B$ a *pairing*. If $B$ is a pairing then, for every triple $(w_i, x_j, y_k) \in M$, either (i) $\{x_j, a_{ijk}\}, \{\bar{a}_{ijk}, y_k\} \in B$ and $\{a_{ijk}, \bar{a}_{ijk}\} \notin B$, or (ii) $\{a_{ijk}, \bar{a}_{ijk}\} \in B$ and $\{x_j, a_{ijk}\}, \{\bar{a}_{ijk}, y_k\} \notin B$. Let $M'$ be the set of triples of type (i). By counting, $|M'| = n$. So there is a 1-1 correspondence between pairings $B$ and sets $M' \subseteq M$ containing $n$ triples. We will now argue that, under this correspondence, bridge-connected graphs $(V, T \cup B)$ are associated with solutions to #3-d Matching and vice versa.

On the one hand, if $M'$ covers all of $W \cup X \cup Y$, then it is easy to check that every edge in $(V, T \cup B)$ is contained in a simple cycle of the form $(r, x_j, a_{ijk}, w_i, r)$ or $(r, y_k, \bar{a}_{ijk}, w_i, r)$ for some triple $(w_i, x_j, y_k) \in M'$, or a cycle of the form $(w_i, a_{ijk}, \bar{a}_{ijk}, w_i)$ for some triple $(w_i, x_j, y_k) \in M \setminus M'$, Conversely, if $(V, T \cup B)$ is bridge connected then in particular $B$ is a pairing, which immediately implies that every element of $X$ and $Y$ is covered by some triple in $M'$. But also every $w_i$ must be covered, since the only way to avoid $\{r, w_i\}$ being a bridge is to have either $\{a_{ijk}, x_j\} \in B$ or $\{\bar{a}_{ijk}, y_k\} \in B$, for some $j, k$ (and hence, in fact, both). This is Observation 3.

Finally to Observation 2. If $(V, T \cup B)$ is not bridge connected then it has a bridge $e$ which is necessarily a tree edge. (The link edges join leaves of the tree $(V,T)$, and hence cannot be bridges.) We have already seen that the existence of a bridge implies $h(B, -q) = 0$. So suppose $(V, T \cup B)$ is bridge connected, and let $M'$ be the corresponding 3-d matching. Then graph $H = G \setminus \overline{B}/B$ may be described as follows.

For each triple $t = (w_i, y_j, z_k) \in M'$, denote by $H_t = (V_t, E_t)$ the graph with vertex set

$$V_t = \{r, w_i, x_j, y_k\} \cup \left\{a_{ij'k'} : (w_i, x_{j'}, y_{k'}) \in M \setminus M'\right\}$$

and edge set

$$E_t = \big\{\{r, w_i\}, \{r, x_j\}, \{r, y_k\}, \{w_i, x_j\}, \{w_i, y_k\}\big\} \\ \cup \big\{\{w_i, a_{ij'k'}\} : (w_i, x_{j'}, y_{k'}) \in M \setminus M'\big\}.$$

The edges with endpoints of the form $a_{ij'k'}$ have multiplicity two, the others multiplicity one. Observe that $V_t \cap V_{t'} = \{r\}$ for distinct triples $t' \neq t$. (This is a consequence of $M'$ being a 3-d matching.) The graph $H$ is obtained by taking the union of all $H_t$ and identifying vertex $r$, so $Z(H; q, \widetilde{w}) = q^{1-n} \prod_t Z(H_t; q, \widetilde{w})$. Each of the multiplicity-two edges ($m - n$ of them) contributes a factor $q(q-1)$, which is non-zero by assumption. That leaves us with $n$ copies of $K_4$ minus an edge. Each of those contributes a factor $-q^4(q-1)(q-2)$, which again is non-zero, by assumption. Putting it all together,

$$Z(H; q, \widetilde{w}) = q^{1-n}[-q^4(q-1)(q-2)]^n [q(q-1)]^{m-n} \\ = (-1)^n q^{2n+m+1}(q-1)^m (q-2)^n.$$

Finally, we need to relate our conveniently chosen weights, $\beta_1$ and $\beta_2$, to the actual ones, $\alpha_1$ and $\alpha_2$. This is done as in the proof of Lemma 13. In particular, we choose $k_1$ and $k_2$ satisfying $(q/\alpha_1 + 1)^{k_1} - 1 \geq 1/\varepsilon$ and $|(q/\alpha_2 + 1)^{k_2}| < \delta$. Now $G'$ is formed as in the proof of Lemma 13 except that $k$-stretches are used in place of $k$-thickenings (according to Equation 14). As before, $k_1 = O(m)$ and $k_2 = O(m^2)$, so the construction is polynomially bounded. □

Using Lemma 15, we can now prove Theorem 3.

**Theorem 3** *Suppose $(x, y) \in \mathbb{Q}^2$ satisfies $q = (x-1)(y-1) \notin \{0, 1, 2\}$. Suppose also that it is possible to shift the point $(x, y)$ to the point $(x', y')$ with $x' \notin [-1, 1]$, and to $(x'', y'')$ with $x'' \in (-1, 1)$. Then there is no FPRAS for* TUTTE$(x, y)$ *unless* RP = NP.

*Proof.* The proof is analogous to the proof of Theorem 2. Note that none of $y$, $y'$ and $y''$ are equal to 1 since $q \neq 0$. Let $\alpha = y - 1$, $\alpha_1 = y' - 1$ and $\alpha_2 = y'' - 1$. The constraints on $x'$ and $x''$, together with $(x'-1)(y'-1) = q$ and $(x''-1)(y''-1) = q$ imply that $x' - 1 = q/\alpha_1 \notin [-2, 0]$ and $x'' - 1 = q/\alpha_2 \in (-2, 0)$. The proof is now exactly the same as the proof of Theorem 2 except that Lemma 15 is used in place of Lemma 13. □

### 4.5 The hyperbola $H_2$ in the halfplane $y < -1$

The following lemma will be used in the proof of Lemma 7.

**Lemma 16.** *Suppose $\alpha_1, \alpha_2 \in \mathbb{Q} - \{0\}$ satisfy $2/\alpha_1 \notin [-2, 0]$ and $2/\alpha_2 \in (-2, 0)$. Then* #PERFECT MATCHINGS $\leq_{\mathrm{AP}}$ MULTITUTTE$(2; \alpha_1, \alpha_2)$

*Proof sketch.* The construction and proof of correctness are simplified versions of those from the proof of Lemma 15, so we provide only a sketch here.

Suppose $G = (V, E)$ is an instance of #PERFECT MATCHINGS. For convenience, set $n = |V|/2$. Let $\widehat{G} = (\widehat{V}, \widehat{E})$ be the graph with vertex set $\widehat{V} = V \cup \{t\}$ and edge set $\widehat{E} = E \cup T$, where $T = \big\{\{t, v\} : v \in V\big\}$. Define $w : \widehat{E} \to \{\beta_1, \beta_2\}$ by $w(e) = \beta_1$ if $e \in E$ and $w(e) = \beta_2$ in $e \in T$. As before, $\beta_1/q$ is small in absolute value, and $\beta_2$ is close to $-q = -2$; specifically, $|\beta_1/q| \le \varepsilon$ and $|1 + 2/\beta_2| \le \delta$.

Following the now familiar path,

$$\begin{aligned} Z(G; 2, w) &= \sum_{A \subseteq \widehat{E}} w(A) 2^{\kappa(A)} \\ &= \sum_{B \subseteq E} \sum_{C \subseteq T} w(B \cup C) 2^{\kappa(B \cup C)} \\ &= \sum_{B \subseteq E} h(B, \beta_2)\, \beta_1^{|B|}, \end{aligned}$$

where

$$h(B; \beta_2) = \sum_{C \subseteq T} \beta_2^{|C|} 2^{\kappa(B \cup C)} \tag{38}$$

Set $Q = q^{n+1}(q-1)^n = 2^{n+1}$. We will show the following observations.

1. If $|B| < n$ then $h(B, -2) = 0$.
2. If $|B| = n$ then

$$h(B, -2) = \begin{cases} Q, & \text{if } (\widehat{V}, T \cup B) \text{ is bridge connected;} \\ 0, & \text{otherwise.} \end{cases}$$

3. The set $\{B : |B| = n \text{ and } (\widehat{V}, T \cup B) \text{ is bridge connected}\}$ is in 1-1 correspondence with the set of solutions to the instance of #PERFECT MATCHINGS. Specifically, $(\widehat{V}, T \cup B)$ is bridge connected iff $B$ is a perfect matching in $G$.

Thus, for $\varepsilon, \delta$ sufficiently small, $\big|2^{-(n+1)} \beta_1^{-n} Z(G; q, w) - N\big| \le \frac{1}{4}$, where $N$ is the number of perfect matchings in $G$. The proof is completed exactly as before.

It remains to justify the three observations. For Observation 1, note that if $|B| < n$ then $(V, B)$ contains an isolated vertex. Consider the factor contributed to $h(B, -2)$ from the edge connecting this vertex to $t$. The contribution is $-q$ (for including the edge) plus $q$ (for excluding it, and hence adding a component), which is 0. Observation 3 is self-evident. Using

Observation 3, we can establish Observation 2 as follows. Suppose that $B$ is a perfect matching. Then

$$h(B,-2) = 2\Big((-2)^2 + 2(-2) + 2\Big)^n = 2^{n+1},$$

where the first 2 comes from the component containing $t$, and for each of the $n$ edges in the matching, the $(-2)^2$ comes from including both edges to $t$, the $2(-2)$ comes from the two ways to add one of the edges to $t$, and the 2 comes from excluding both edges to $t$, which adds a component.

$\square$

We can now prove Lemma 7.

**Lemma 7.** *Suppose* $(x, y)$ *is a point with* $(x-1)(y-1) = 2$ *and* $y < -1$. *Then* $\text{TUTTE}(x, y) \equiv_{\text{AP}} \#\text{PERFECT MATCHINGS}$.

*Proof.* To show $\#\text{PERFECT MATCHINGS} \leq_{\text{AP}} \text{TUTTE}(x, y)$ use thickenings as in the proof of Corollary 5 to shift $(x, y)$ to a point $(x', y')$ with $x' \notin [-1, 1]$ and to a point $(x'', y'')$ with $x'' \in (-1, 1)$. Then follow the proof of Theorem 3 to reduce $\text{MULTITUTTE}(2; \alpha_1, \alpha_2)$ to $\text{TUTTE}(x, y)$. Finally, Lemma 16 reduces $\#\text{PERFECT MATCHINGS}$ to $\text{MULTITUTTE}(2; \alpha_1, \alpha_2)$.

We now show $\text{TUTTE}(x, y) \leq_{\text{AP}} \#\text{PERFECT MATCHINGS}$. Using the Fortuin-Kasteleyn representation of the Potts model, $Z(G; 2, y-1)$ is equal to the partition function of the Ising model in which every edge has weight $y - 1$. That is,

$$Z(G; 2, y-1) = \sum_{\sigma: V(G) \to \{-1, 1\}} y^{\text{mono}(\sigma)}, \tag{39}$$

where $\text{mono}(\sigma)$ denotes the number of monochromatic edges in the mapping $\sigma$. (See [17, (2.7), (2.9)] for a justification of this identity.) We will assume without loss of generality that the graph $G$ has no loops. It is clear from (39) that a loop merely introduces a factor of $y$.

Let $n = |V(G)|$ and $m = |E(G)|$. Let $\nu = (y-1)/(y+1)$. Now let $G'$ be the graph constructed from $G$ by replacing each vertex of degree $\ell \geq 4$ as follows. Suppose that the neighbours of vertex $v$ in $G$ are $w_1, \ldots, w_\ell$. Then replace $v$ with a path of $\ell - 2$ new degree-3 vertices $v_2, \ldots, v_{\ell-1}$. The edges $(v_2, v_3), (v_3, v_4), \ldots, (v_{\ell-2}, v_{\ell-1})$ will be called "supplementary" edges of $G'$. The edges $(w_1, v), (w_2, v), \ldots, (w_{\ell-1}, v), (w_\ell, v)$ correspond to edges $(w_1, v_2), (w_2, v_2), (w_3, v_3), \ldots, (w_{\ell-1}, v_{\ell-1}), (w_\ell, v_{\ell-1})$ of $G'$. We will call these edges "primary", because they correspond to the original edges of $G$.

Then let $G''$ be the graph constructed from $G'$ by replacing each vertex of degree 2 as follows. Suppose that the neighbours of vertex $v$ in $G'$ are $w_1$ and $w_2$. Then replace $v$ with two new vertices $v_1$ and $v_2$ and replace the edges $(w_1, v)$ and $(w_2, v)$ with the path $(w_1, v_1), (v_1, v_2), (v_2, w_2)$ in which $(v_1, v_2)$

is a supplementary edge of $G''$ and the edges $(w_1, v_1)$ and $(v_2, w_2)$ are "primary" edges of $G''$. Finally, if $v$ is a degree-3 vertex of $G'$ with neighbours $w_1$, $w_2$ and $w_3$, replace $v$ with the three vertices $v_1, v_2, v_3$. Add supplementary edges $(v_1, v_2), (v_2, v_3), (v_3, v_1)$. Replace the edges $(w_1, v)$, $(w_2, v)$ and $(w_3, v)$ of $G'$ with edge $(w_1, v_1)$, $(w_2, v_2)$ and $(w_3, v_3)$ of $G''$, making edge $(w_i, v_i)$ primary if and only if $(w_i, v)$ was primary in $G'$. (Once again, the primary edges of $G''$ correspond to the original edges of $G$.)

Fisher has shown [7, (10)] that

$$Z(G; 2, y-1) = y^m 2^n \left(\frac{\nu}{1+\nu}\right)^m \sum_X \prod_e \frac{1}{\nu}, \tag{40}$$

where the sum is over perfect matchings $X$ of $G''$ and the product is over primary edges $e$ of $G''$ that are in the perfect matching $X$.

Now let $n_1$ and $n_2$ be positive integers such that $1/\nu = n_1/n_2$. Let $H$ be a graph consisting of $n_1$ parallel edges from a vertex $u$ to a vertex $a$ and $n_2$ parallel edges from the vertex $a$ to a vertex $b$ and a single edge from $b$ to a vertex $v$. Let $\mathcal{M}$ be the set of matchings of $H$ which match both $a$ and $b$. There are $n_1$ matchings in $\mathcal{M}$ in which $a$ is matched with $u$. All of these match both $u$ and $v$. There are $n_2$ matchings in $\mathcal{M}$ in which $a$ is matched with $b$. These do not match $u$ or $v$. There are no other matchings in $\mathcal{M}$.

Construct $\widehat{G}$ from $G''$ by replacing every primary edge $(u, v)$ of $G''$ with a copy of $H$. Then the expression $\sum_X \prod_e \frac{1}{\nu}$ in Equation (40) is equal to the number of perfect matchings of $\widehat{G}$ divided by $n_2^m$. So if we could approximate the number of perfect matchings of $\widehat{G}$, we could approximate $Z(G; 2, y-1)$. $\square$

**Remark.** The construction in the reduction from the problem TUTTE$(x, y)$ to the problem #PERFECT MATCHINGS relies on the fact that $y-1$ and $y+1$ have the same sign (so $n_1$ and $n_2$ are both positive integers). The same reduction would apply for $q = 2$ and $y > 1$ but this is ferromagnetic Ising, and we already have an FPRAS, due to Jerrum and Sinclair [12].

# 5 #P-hardness

In Section 2.3, we noted that if $x \geq 0$ and $y \geq 0$ then TUTTE$(x, y)$ is in $\#\mathrm{P}_{\mathbb{Q}}$, so there is a randomised approximation scheme for TUTTE$(x, y)$ that runs in polynomial time using an oracle for an NP predicate. Here we show that it is unlikely that TUTTE$(x, y)$ is in $\#\mathrm{P}_{\mathbb{Q}}$ for all $x$ and $y$. In particular, we identify a region of points $(x, y)$ where $y$ is negative for which even approximating TUTTE$(x, y)$ is as hard as #P. Specifically, we prove the following.

**Theorem 17.** *Suppose $(x, y)$ is a point with $y \in (-1, 0)$ and $(x-1)(y-1) = 4$. Then there is no FPRAS for* TUTTE$(x, y)$ *unless* RP = #P.

## 5.1 The Potts model

For a positive integer $q$ and a $y \in \mathbb{Q}$, and a graph $G = (V, E)$, let

$$P(G; q, y) = \sum_{\sigma: V \to \{1,\ldots,q\}} y^{\mathrm{mono}(\sigma)},$$

where $\mathrm{mono}(\sigma)$ is the number of edges in $E$ that are monochromatic under the map $\sigma$. $P(G; q, y)$ is the partition function of the *$q$-state Potts model* at an appropriate temperature (depending on $y$). The region $y \geq 1$ is "ferromagnetic" since like spins are favoured along an edge, the region $0 \leq y \leq 1$ is "antiferromagnetic", and the region $y \leq 0$ is "unphysical" [17]. It is known that the $q$-state Potts model coincides with the Tutte polynomial when $q$ is a positive integer. In particular (see (10) and [17, (2.9)]),

$$T(G; x, y) = (y-1)^{-n}(x-1)^{-\kappa(E)} P(G; q, y),$$

where $q = (x-1)(y-1)$.

In the rest of this section, we suppose that we have an FPRAS for $P(G; 4, y)$ for a point $y \in (-1, 0)$ and we show how to use the FPRAS to solve a #P-hard problem (counting proper 3-colourings of a simple graph).

First, we establish some notation. If $G$ is a graph with designated vertices $a$ and $b$ and $\alpha$ and $\beta$ are values in $\{1, \ldots, q\}$, let $P(G; q, y \mid \sigma(ab) = \alpha\beta)$ denote the contribution to $P(G; q, y)$ due to colourings $\sigma$ with $\sigma(a) = \alpha$ and $\sigma(b) = \beta$.

## 5.2 The building blocks

Fix $y \in (-1, 0)$. Suppose that $n$ is the number of vertices of a graph $G$. Let $M$ be a rational number in the range $1 \leq M \leq 3^n$ and let $\varepsilon = 2^{-n^2}$. In this section, we will show how to construct a graph $H_M$ with two designated vertices, $a$ and $b$, so that

$$\frac{-1}{M} \leq \frac{P(H_M; 4, y \mid \sigma(ab) = 11)}{P(H_M; 4, y \mid \sigma(ab) = 12)} \leq \frac{-1}{M} + \varepsilon. \tag{41}$$

As a building block, let $P_\ell$ be an $\ell$-edge path. Let $f_\ell$ denote $P(P_\ell; 4, y \mid \sigma(ab) = 11)$ and let $a_\ell$ denote $P(P_\ell; 4, y \mid \sigma(ab) = 12)$. These satisfy the recurrences $f_\ell = y f_{\ell-1} + 3a_{\ell-1}$ and $a_\ell = f_{\ell-1} + (2+y)a_{\ell-1}$ with $f_1 = y$ and $a_1 = 1$. The solution to these recurrences, for $\ell \geq 1$, is given by

$$f_\ell = \frac{1}{4}(3+y)^\ell + \frac{3}{4}(y-1)^\ell,$$

and

$$a_\ell = \frac{1}{4}(3+y)^\ell - \frac{1}{4}(y-1)^\ell.$$

Thus,

$$\frac{a_\ell}{f_\ell} = \frac{(3+y)^\ell - (y-1)^\ell}{(3+y)^\ell + 3(y-1)^\ell} = 1 - \frac{4(y-1)^\ell}{(3+y)^\ell + 3(y-1)^\ell}.$$

Recall $y > -1$ and let $\gamma = ((3+y)/(y-1))^2 > 1$. For every positive integer $j$, let $\delta_j = 1 - a_{2j}/f_{2j}$. Note that

$$0 < \gamma^{-j} < \delta_j < 4\gamma^{-j}. \tag{42}$$

Also, $f_{2j}/a_{2j} = 1/(1-\delta_j)$.

Given $y$, choose a positive odd integer $k$ so that

$$|y|^k \leq \frac{1}{M} < |y|^{k-2}. \tag{43}$$

Now, let $t$ be the smallest integer such that $\delta_t \leq \varepsilon M$. For $j \in \{1, \ldots, t\}$, choose a natural number $k_j$ so that

$$|y|^k \prod_{r=1}^{j-1} \frac{1}{(1-\delta_r)^{k_r}} \frac{1}{(1-\delta_j)^{k_j}} \leq \frac{1}{M} < |y|^k \prod_{r=1}^{j-1} \frac{1}{(1-\delta_r)^{k_r}} \frac{1}{(1-\delta_j)^{k_j+1}}. \tag{44}$$

Now $H_M$ is formed by joining a number of paths, all with endpoints $a$ and $b$. To form $H_M$, take $k$ paths of length 1 (i.e., edges). Also, for every $j \in \{1, \ldots, t\}$, take $k_j$ paths of length $2j$. So

$$\frac{P(H_M; 4, y \mid \sigma(ab) = 11)}{P(H_M; 4, y \mid \sigma(ab) = 12)} = -|y|^k \prod_{r=1}^{t} \frac{1}{(1-\delta_r)^{k_r}},$$

and this is at least $-1/M$ and at most $-(1/M)(1-\delta_t)$, which implies Equation (41).

Now we consider the size of $H_M$. Equation (43) implies that $k = O(\log M) = O(n)$. Also, $t = O(n^2)$ by (42).

How big can $k_j$ be? By (44) we have $\frac{1}{(1-\delta_j)^{k_j}} \leq \frac{1}{1-\delta_{j-1}}$, so $(1-\delta_j)^{k_j} \geq 1 - \delta_{j-1}$. $\delta_j$ is decreasing in $j$, so without loss of generality, we'll deal with those $j$ such that $\delta_j \leq 0.7$ (the values of $k_j$ corresponding to smaller values of $j$ are just constants). Then

$$(1-\delta_j)^{2/\delta_j} < 0.15 < (1-\delta_{j-1})^{1/\delta_{j-1}},$$

so

$$(1-\delta_j)^{2\delta_{j-1}/\delta_j} < (1-\delta_{j-1})^{\delta_{j-1}/\delta_{j-1}} = 1 - \delta_{j-1},$$

and therefore $k_j \leq 2\delta_{j-1}/\delta_j = O(1)$.

### 5.3 The construction

We use the notation from Section 5.2. Let $r$ be the smallest even integer such that $|y|^r < \varepsilon 4^{-n}$. Construct $G'$ from the simple graph $G$ (the graph we wish to 3-colour) by replacing every edge $(u, v)$ of $G$ with a bundle of $r$ parallel edges with endpoints $u$ and $v$. (That is, we perform an $r$-thickening on all edges.) Add two new vertices, $a$ and $b$. Connect $a$ to every vertex in $G$ by a bundle of $r$ parallel edges. Similarly, connect $b$ to every vertex in $G$.

Let $n$ denote the number of vertices of $G$ and $m$ denote the number of edges of $G$. Recall that $P(G; 3, 0)$ is the number of proper 3-colourings of $G$. Then,

$$P(G; 3, 0) \leq P(G'; 4, y \mid \sigma(ab) = 11) \leq P(G; 3, 0) + 4^n |y|^r,$$

so

$$P(G; 3, 0) \leq P(G'; 4, y \mid \sigma(ab) = 11) \leq P(G; 3, 0) + \varepsilon. \tag{45}$$

Similarly,

$$P(G; 2, 0) \leq P(G'; 4, y \mid \sigma(ab) = 12) \leq P(G; 2, 0) + \varepsilon. \tag{46}$$

Let $G_M$ be the graph constructed from $G'$ and $H_M$ by identifying vertex $a$ in $G'$ with vertex $a$ in $H_M$ and similarly identifying vertex $b$ in $G'$ with vertex $b$ in $H_M$. Let

$$Y_M = \frac{P(G_M; 4, y)}{P(H_M; 4, y \mid \sigma(ab) = 12)}$$

and note (from the previous section) that the quantity $P(H_M; 4, y \mid \sigma(ab) = 12)$ in the denominator is positive. Now

$$P(G_M; 4, y \mid \sigma(ab) = 11) = P(H_M; 4, y \mid \sigma(ab) = 11) P(G'; 4, y \mid \sigma(ab) = 11),$$

and

$$P(G_M; 4, y) = 4P(G_M; 4, y \mid \sigma(ab) = 11) + 12P(G_M; 4, y \mid \sigma(ab) = 12),$$

so

$$\begin{aligned} Y_M = 4\, &\frac{P(H_M; 4, y \mid \sigma(ab) = 11)}{P(H_M; 4, y \mid \sigma(ab) = 12)}\, P(G'; 4, y \mid \sigma(ab) = 11) \\ &+ 12P(G'; 4, y \mid \sigma(ab) = 12). \end{aligned}$$

Let $\xi = 5\varepsilon 3^n = o(1)$. By Equations (41), (45), and (46),

$$Y_M = -4\, \frac{P(G; 3, 0)}{M} + 12P(G; 2, 0) + \varrho_M,$$

where $|\varrho_M| \leq \xi$.

We will restrict attention to graphs $G$ which are bipartite with at least 4 vertices. Note that it is #P-hard to count the proper 3-colourings of a bipartite graph. For example, [4, Section 6] observes that this is the same as counting homomorphisms from a general graph to the cycle $C_6$, which is shown to be #P-hard by Dyer and Greenhill [6]. Also, for such a graph $G$, $P(G;2,0) > 0$ and $P(G;3,0) \geq 4P(G;2,0)$.

Now, suppose that we had an FPRAS for approximating $P(G_M;4,y)$. A call to the FPRAS gives us the sign of $Y_M$.

Let $G$ be a bipartite graph with $n \geq 4$ vertices. Let $z_\ell = 3^{-n}$ and $z_u = 1$. Then we have an interval $[z_\ell, z_u]$ with $Y_{1/z_\ell} > 0$ and $Y_{1/z_u} < 0$. Use binary search to bisect the interval until we have an interval $[z_\ell, z_u]$ with $Y_{1/z_\ell} \geq 0$, $Y_{1/z_u} \leq 0$, and $z_u - z_\ell \leq \varepsilon$. (This takes at most $n^2$ bisections since, after $j$ bisections, $z_u - z_\ell \leq 2^{-j}$.)

Since $Y_{1/z_\ell} \geq 0$, we have

$$z_\ell \leq \frac{3P(G;2,0)}{P(G;3,0)} + \frac{\xi}{4P(G;3,0)}.$$

Similarly, since $Y_{1/z_u} \leq 0$,

$$z_u \geq \frac{3P(G;2,0)}{P(G;3,0)} - \frac{\xi}{4P(G;3,0)}.$$

So

$$\frac{3P(G;2,0)}{P(G;3,0)} - \frac{\xi}{4P(G;3,0)} \leq z_u \leq z_\ell + \varepsilon \leq \frac{3P(G;2,0)}{P(G;3,0)} + \frac{\xi}{4P(G;3,0)} + \varepsilon.$$

Now the point is that only one real number in the specified interval for $z_u$ is of the form $3n_1/n_2$ where $n_1$ is an integer in $\{1,\dots,2^n\}$ and $n_2$ is an integer in $\{1,\dots,3^n\}$ (since $\varepsilon$ and $\xi$ are so small) so the value of $z_u$ allows us to compute $3P(G;2,0)/P(G;3,0)$ exactly, and since $P(G;2,0)$ can be computed exactly, this gives us $P(G;3,0)$, thus counting proper 3-colourings of $G$.

## References


[1] Noga Alon, Alan Frieze, and Dominic Welsh. Polynomial time randomized approximation schemes for Tutte-Gröthendieck invariants: the dense case. *Random Struct. Algorithms*, 6(4):459–478, 1995.

[2] Thomas Brylawski. The Tutte polynomial. I. General theory. In *Matroid theory and its applications*, pages 125–275. Liguori, Naples, 1982.

[3] E. Dahlhaus, D. S. Johnson, C. H. Papadimitriou, P. D. Seymour, and M. Yannakakis. The complexity of multiterminal cuts. *SIAM J. Comput.*, 23(4):864–894, 1994.

[4] Martin Dyer, Leslie Ann Goldberg, Catherine Greenhill, and Mark Jerrum. The relative complexity of approximate counting problems. *Algorithmica*, 38(3):471–500, 2004.

[5] Martin E. Dyer, Leslie Ann Goldberg, Catherine S. Greenhill, and Mark Jerrum. The relative complexity of approximate counting problems. *Algorithmica*, 38(3):471–500, 2003.

[6] Martin E. Dyer and Catherine S. Greenhill. The complexity of counting graph homomorphisms. *Random Structures and Algorithms*, 17(3-4):260–289, 2000.

[7] Michael E. Fisher. On the dimer solution of planar ising models. *Journal of Mathematical Physics*, 7(10):1776–1781, 1966.

[8] Greg N. Frederickson and Joseph Ja'Ja'. Approximation algorithms for several graph augmentation problems. *SIAM J. Comput.*, 10(2):270–283, 1981.

[9] Leslie Ann Goldberg and Mark Jerrum. Inapproximability of the tutte polynomial. In *STOC '07: Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 459–468, New York, NY, USA, 2007. ACM Press.

[10] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Cambridge Philos. Soc.*, 108(1):35–53, 1990.

[11] M. Jerrum. *Counting, sampling and integration: algorithms and complexity*. Birkhauser Boston, also see author's website, 2003.

[12] Mark Jerrum and Alistair Sinclair. Polynomial-time approximation algorithms for the Ising model. *SIAM J. Comput.*, 22(5):1087–1116, 1993.

[13] Mark R. Jerrum, Leslie G. Valiant, and Vijay V. Vazirani. Random generation of combinatorial structures from a uniform distribution. *Theoret. Comput. Sci.*, 43(2-3):169–188, 1986.

[14] David R. Karger. A randomized fully polynomial time approximation scheme for the all-terminal network reliability problem. *SIAM Rev.*, 43(3):499–522, 2001.

[15] Nathan Linial. Hard enumeration problems in geometry and combinatorics. *SIAM J. Algebraic Discrete Methods*, 7(2):331–335, 1986.

[16] P. D. Seymour. Nowhere-zero 6-flows. *J. Combin. Theory Ser. B*, 30(2):130–135, 1981.

[17] Alan Sokal. The multivariate Tutte polynomial. In *Surveys in Combinatorics*. Cambridge University Press, 2005.

[18] Larry Stockmeyer. The complexity of approximate counting. In *STOC '83: Proceedings of the fifteenth annual ACM symposium on Theory of computing*, pages 118–126, New York, NY, USA, 1983. ACM Press.

[19] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Canadian J. Math.*, 6:80–91, 1954.

[20] W. T. Tutte. *Graph theory*, volume 21 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1984. With a foreword by C. St. J. A. Nash-Williams.

[21] L. G. Valiant and V. V. Vazirani. NP is as easy as detecting unique solutions. *Theor. Comput. Sci.*, 47(1):85–93, 1986.

[22] D. J. A. Welsh. *Complexity: knots, colourings and counting*, volume 186 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

[23] D. J. A. Welsh. Randomised approximation in the Tutte plane. *Combin. Probab. Comput.*, 3(1):137–143, 1994.

[24] David Zuckerman. On unapproximable versions of NP-Complete problems. *SIAM Journal on Computing*, 25(6):1293–1304, 1996.